January 26, 2019

CECH COHOMOLOGY AND GOOD COVERS

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ABSTRACT. Given a topological space X, we can compute the Cech cohomology of X with respect to some cover $\mathcal{U}$. By taking a refinement of $\mathcal{U}$, $\mathcal{V}$, we have a map

$$\coprod_{V\in\mathcal{V}} \hookrightarrow \coprod_{U\in\mathcal{U}}$$

which is a map between chain complexes and therefore gives us a map on cohomology: $\check{H}^\bullet(X,\mathcal{V}) \leftarrow \check{H}^\bullet(X,\mathcal{U})$. In other words, when we use a refinement of some other cover, the refinement may yield different answers in its cohomology groups. To obtain a definite answer for the Cech cohomology, we must utilize direct limits. However, we can avoid using direct limits with the use of a theorem which states that if use a good cover, then any refinement of that cover will have an isomorphism on its cohomology groups. Once we have defined Cech cohomology, we show that there exists an algorithm that can compute the Cech cohomology for us. The algorithm needs to use ϵ-cubes instead of ϵ-balls, as it is difficult to determine intersections of multiple ϵ-balls algorithmically.

CONTENTS

Before we understand what Cech cohomology is, let us first provide some motivation for learning about it. In layman's terms, Cech cohomology tells us about the structure of a space. In particular, we can learn about certain features, such as of holes or loops, present in the set. This idea is not just interesting to those studying pure mathematics, but also to those in applied mathematics. For example, an algorithm that computes the Cech cohomology on a data set can tell us if there are any "gaps" in the data.

To understand Cech cohomology, and how the process works, we must gain a working vocabulary in topology, algebra (especially homological algebra), and category theory. Only then can we understand the formal definition of Cech cohomology. Once we understand Cech cohomology, we can outline an algorithm that performs the Cech cohomology on a data set. Lastly, we will compare the Cech cohomology to the Vietoris-Rips Complex.

1. Topology

To understand Cech cohomology, an elementary knowledge of topology is required. Basic ideas of set theory are also required, however it will be assumed that the reader is already familiar with such concepts. Our foray into topology will begin, of course, with the definition of a topology.

Definition 1.1. (Topology) Let X be a set. A topology on X is a collection of subsets of X denoted by τ and satisfying the following three properties:

1. X and $\emptyset$ are elements of τ
2. Arbitrary unions of elements of τ are also elements τ.
3. Finite intersections of elements of τ are also elements of τ

Any set which can have a topology defined is said to be a *topological space*, and any element of a topology is said to be an *open set* on the topological space. (Munkres, 74).

Interestingly, having a topology is not a strong requirement. This is because in general we can define a topology commonly referred to as the *trivial topology* on a set X. The aforementioned topology, τ, is just the set containing X and $\emptyset$. One sees that this is indeed a topology, since $X \cup \emptyset = X$ and $X \cap \emptyset = \emptyset$ and thus the three requirements of a topology are satisfied. Many of the sets that are of particular interest to mathematicians are also topological spaces.

Example 1.2. (A topology on $\mathbb{R}^n$) We can show that the set $\mathbb{R}^n$ has a topology. The topology we will use in this example is referred to as the *usual topology* on $\mathbb{R}^n$. To define our topology, we use the euclidean distance formula on $\mathbb{R}^n$, given by $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$. Choosing a distance $\epsilon > 0$, we can define a set $B_\epsilon(x) = \{y \in \mathbb{R}^n | d(x, y) < \epsilon$ for some point $x \in \mathbb{R}^n\}$. Note that the sets of $B_\epsilon(x)$ are just disks in $\mathbb{R}^n$ with radius ϵ centered at X. Now we define the standard topology on $\mathbb{R}^n$ by $\tau = \{U \subseteq \mathbb{R}^n |$ for all $u \in U$, there exists some $\epsilon > 0$ such that $B_\epsilon(u) \subseteq U\}$. In other words, a set is open if every point can be surrounded by a disk which is contained inside the set. We can show that τ is a topology on $\mathbb{R}^n$. First, we show that $\mathbb{R}^n$ and $\emptyset$ are both elements of τ. It is easy to see that $\mathbb{R}^n \in \tau$. Any point $x \in \mathbb{R}^n$ can be surrounded by a ball contained entirely within $\mathbb{R}^n$, and so we say $\mathbb{R}^n \in \tau$. Next we check that $\emptyset \in \tau$. There is no element of $\emptyset$ for us to surround with an ϵ-ball, and so it is vacuously true that $\emptyset \in \tau$.

Next we check that an arbitrary union of elements of τ are also elements of τ. We consider the arbitrary union

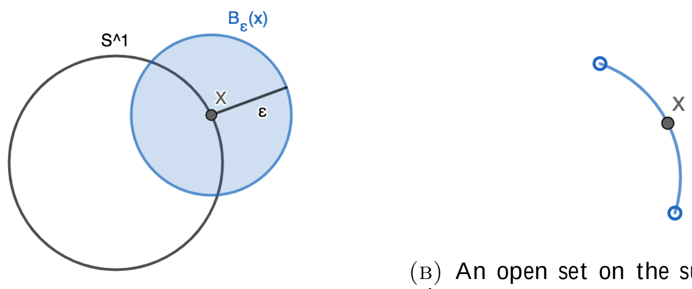
$$\bigcup_{j \in J} U_j$$

where each $U_j \in \tau$. Consequently, each $u_j \in U_j$ can be surrounded by some ϵ-ball in U_j. Then if x is in the arbitrary union, it follows that $x \in U_1$ or $x \in U_2$ or $x \in U_3$ and so on. We know then that x is in some U_j, and as we know any element of U_j can be surrounded by an ϵ-ball. Since this U_j is contained in the arbitrary union, and the ϵ-ball is contained within U_j, we have that

$$B_\epsilon(x) \subseteq \bigcup_{j \in J} U_j \text{ for all } x \in \bigcup_{j \in J} U_j \text{ which implies } \bigcup_{j \in J} U_j \in \tau$$

Lastly, we show that any finite intersection

$$\bigcap_{j=1}^{k} U_j$$



(A) S^1 (shown in black) with an open set of $\mathbb{R}^2$ (blue).

(B) An open set on the subspace topology of S^1 resulting from 1A.

FIGURE 1. S^1 with an open set of $\mathbb{R}^2$ (A) and an open set of the subspace topology on S^1 (B).

of elements of τ are also elements of τ. Let x be an element of

$$\bigcap_{j=1}^{k} U_j$$

Then $x \in U_j$ for all $j \in \{1, \ldots, k\}$. Since each $U_j \in \tau$, x can be surrounded by an ϵ-ball contained within each U_j. Our intersection consists of a finite number of subsets of $\mathbb{R}^n$, and as such, there exists a smallest ϵ-ball containing x. It follows that for the smallest ϵ-ball, denoted by $B_\epsilon(x)$, we have $B_\epsilon(x) \subseteq B_{\epsilon'}(x)$ for all $B_{\epsilon'}(x) \subseteq U_j$ where U_j is any set in the finite intersection. Thus

$$B_\epsilon(x) \subseteq \bigcap_{j=1}^{k} U_j$$

and

$$\bigcap_{j=1}^{k} U_j \in \tau.$$

Therefore, the topology generated by ϵ-balls is a topology on $\mathbb{R}^n$.

As we have just shown, $\mathbb{R}^n$ is a topological space. This result is particularly useful for applied mathematics, where most work is contained in some subset of $\mathbb{R}^n$. Importantly, it turns out that we can define a topology on any subset of $\mathbb{R}^n$, which allows us to extend any properties of topological spaces to said subset.

Definition 1.3. (Subspace topology) Let X be a topological space with a topology τ_X and $Y \subseteq X$. We can define a topology on Y, called the subspace topology, by $\tau_Y = \{U \cap Y | U \in \tau_X\}$ (Munkres, 86).

Example 1.4. (Subspace topology on S^1) Consider $S^1 \subseteq \mathbb{R}^2$. Recall from Example 1.2 that open sets on $\mathbb{R}^2$ are in the form of disks with radius $\epsilon > 0$. Open sets on S^1, then, are points on S^1 intersected with these ϵ-balls. In other words, open sets of S^1 are just arcs, as you can see in Figure 1.

Example 1.4 hints at an important concept in our discussion of Cech cohomology. Notice that an open set on the subspace topology of S^1 was just a "fragment" of S^1. It follows that if we used enough carefully placed ϵ-balls, the resulting open sets in the subspace topology would "cover" S^1.

Definition 1.5 (Open cover)**.** Let X be a topological space, and let $\mathcal{U}$ be a collection of subsets of X. We say that $\mathcal{U} = \{U_i\}_{i \in I}$, is a *cover* of X if $\bigcup_{i \in I} U_i = X$. If each set U_i of $\mathcal{U}$ is *open* on X, then $\mathcal{U}$ is called an *open cover* of X.

Covers are the cornerstone of computing the Cech cohomology. We make use of covers to induce a *Cech Complex* on a given topological space, which allows us to compute the necessary *cohomology groups*. Open covers are required for this process, as we will discuss later on. For now, we familiarize ourselves with open covers through the following examples.

Example 1.6 (An open cover of S^1)**.** One way to create an open cover is to obtain one from the subspace topology. Consider $S^1 \subset \mathbb{R}^2$. We know from Example 1.2 that open sets of $\mathbb{R}^2$ are disks with some radius ϵ. Then we can cover S^1 with these ϵ-balls; in this case, we use three. However, our goal is to cover S^1 with sets that are open in S^1, and ϵ-balls alone do not meet the requirement. However, we can intersect each of our ϵ-balls with S^1 and obtain open sets on the subspace topology for S^1. In doing so, we obtain three open sets whose union is equal to S^1.

Example 1.7 (An open cover of $\mathbb{R}^n$)**.** We can create an open cover of $\mathbb{R}^n$ using ϵ-balls. Let $U = \{B_\epsilon(x) | x \in \mathbb{R}^n\}$ That is, let U be the set of ϵ-balls centered at some point $x \in \mathbb{R}^n$. We can consider each $B_\epsilon(x) \subset \mathbb{R}^n$, and since each ϵ-ball is contained within itself, it follows that each $B_\epsilon(x)$ is open on $\mathbb{R}$. Lastly, note that each $x \in \mathbb{R}^n$ can be surrounded by some ϵ-ball, and so if we take the union of all the ϵ-balls - that is, the union of all the elements of U - we find that the union necessarily yields $\mathbb{R}^n$. We have now shown that U creates an open cover of $\mathbb{R}^n$.

Note that this cover is uncountably infinite. There is no limit to how many elements are in a cover. The only requirement is that the union of every set in the cover is equal to the original space.

We are not limited to using ϵ-balls to create an open cover of $\mathbb{R}^n$. As long as the sets in our open cover are in some topology on $\mathbb{R}^n$, we can make them whatever shape we wish. An open cover of cubes will be particularly useful for those wishing to make an algorithm for the Cech cohomology.

Example 1.8. (A cover for $\mathbb{R}^n$ with cubes) If we wish to cover $\mathbb{R}^n$, we are not limited to just ϵ-balls. We can just as easily use ϵ-cubes: n-dimensional cubes with side length ϵ. Let $U = \{S_\epsilon(x) | x \in \mathbb{R}^n\}$, where $S_\epsilon(x) = \{y \in \mathbb{R}^n$ such that $|y_i - x_i| < \epsilon/2$ for all $i \in \{1, \ldots, n\}, x \in \mathbb{R}^n\}$. The set $S_\epsilon(x)$ is just a cube centered at a point $x \in \mathbb{R}^n$ whose side length is ϵ, and the set U is the collection of all such cubes for every point in $\mathbb{R}^n$. Now if τ is the topology generated by surrounding every point in some $U \subset \mathbb{R}^n$ with an ϵ-cube contained within U, we find that each $S_\epsilon(x)$ is open in τ. If we take the union of each element of U, we take the union of all the ϵ-cubes centered at each point of $\mathbb{R}^n$, which of course is also $\mathbb{R}^n$. Thus, we can create a cover of $\mathbb{R}^n$ by using cubes.

To show that this forms an open cover, we have to show that the cubes are open. There are two ways to accomplish this:

(1.) Show that under the usual topology, each ϵ-cube sits inside an ϵ-ball. Since each ϵ-ball is an open set in the usual topology, it follows that an ϵ-cube contained in an ϵ-ball will also be open.

(2.) Show that there is a topology generated by ϵ-cubes, constructed the same way as the usual topology. It follows from the definition (once we prove this is indeed a topology) that each ϵ-cube is open, since every element of an ϵ-cube is obviously contained by an ϵ-cube inside the same set.

For the sake of brevity and originality, we take the first option, since the second choice would be a proof nearly identical to the one in Example 1.2. Consider $\mathbb{R}^n$ under the usual topology, and let $S_\epsilon(s)$ be an ϵ-cube and $x \in S_\epsilon(s)$. Then $|x_i - s_i| < \epsilon/2$ for all $i \in \{1, \ldots, n\}$. It follows that $|x_i - s_i| \geq 0$ and $\epsilon > 0$, and so we also have the inequality $|x_i - s_i|^2 < (\epsilon/2)^2 = \epsilon^2/4$ for all $i \in \{1, \ldots, n\}$. Since each $|x_i - s_i| \in \mathbb{R}$, it follows that $|x_i - s_i|^2 = (x_i - s_i)^2$ for all $i \in \{1, \ldots, n\}$. Then our equality $|x_i - s_i|^2 < \epsilon^2/4$ becomes $(x_i - s_i)^2 < \epsilon^2/4$. It follows then that

$$\sum_{i=1}^{n} (x_i - s_i)^2 = (x_1 - s_1)^2 + \cdots + (x_n - s_n)^2 < \epsilon^2/4 + \epsilon^2/4 + \cdots + \epsilon^2/4 = \Rightarrow d(x, s)^2 < n\epsilon^2/4$$

Since $d(x, s)^2 \geq 0$ and $n\epsilon^2/4 > 0$, we have that $d(x, s) = d(s, x) < \sqrt{n}\epsilon/2$. Thus, $x \in B_{\sqrt{n}\epsilon/2}(s)$ and $S_\epsilon(s) \subset B_{\sqrt{n}\epsilon/2}(s)$. This implies $S_\epsilon(s)$ is an open set of the usual topology. Any element in $B_{\sqrt{n}\epsilon/2}(s)$ can be surrounded by an δ-ball $B_{\epsilon'}(s) \subset B_{\sqrt{n}\epsilon/2}(s)$. Since any element of $S_\epsilon(s)$ belongs to $B_{\sqrt{n}\epsilon/2}(s)$, it follows that any element of $S_\epsilon(s)$ can be surrounded by an ϵ-ball small enough to fit in the cube. Thus, each set in the cover U is open, as desired.

We see quite clearly that there are often multiple open covers for a topological space. We could cover $\mathbb{R}^n$ with infinitely-many ϵ-balls, or we could cover $\mathbb{R}^n$ with infinitely-many $\epsilon/2$-balls. Intuitively, we know that each $\epsilon/2$-ball will be smaller, and will thus sit inside some ϵ-ball. It should follow then that we can map all of the smaller $\epsilon/2$-balls to their supersets, the ϵ-balls.

Definition 1.9. (Refinement) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of a topological space X. A refinement is a cover $\mathcal{V} = \{V_j\}_{j \in J}$ of X such that for all $j \in J$ there exists an $i \in I$ such that $V_j \subseteq U_i$. Importantly, this means that there exists a refinement map

$$\coprod_{V \in \mathcal{V}} V \hookrightarrow \coprod_{U \in \mathcal{U}} U$$

Refinements are important to the formal definition of Cech cohomology. We will need to take the "most refined" open cover for a topological space as part of the definition. Refinements are also a particularly interesting aspect of *sheaves*, another cornerstone of Cech cohomology, which we will discuss in Section 4. For now, we look at a refinement of ϵ-balls.

Example 1.10. (Refinement of a cover of $\mathbb{R}^n$ with ϵ-balls) Consider the cover $\mathcal{U} = \{B_\epsilon(x) | x \in \mathbb{R}^n\}$. Recall that each ball $B_\epsilon(x)$ is a disk of radius $\epsilon > 0$. Then the ball $B_{\epsilon/2}(x) \subseteq B_\epsilon(x)$. If we let $\mathcal{V} = \{B_{\epsilon/2}(x) | x \in \mathbb{R}^n\}$, then for each open set $V \in \mathcal{V}$ we have an open set $U \in \mathcal{U}$ such that $V \subseteq U$. Then $\mathcal{V}$ is a refinement of the open cover $\mathcal{U}$.

Topologists are especially interested in the continuous functions between topological spaces. However, the usual idea of continuity that is taught in real analysis and calculus only applies to sets like $\mathbb{R}^n$, which have notions of distance. Not every topological space has the property of distance, and as such a different definition of continuity is needed.

Definition 1.11. (Continuous functions) Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X. For clarification, note that by $f^{-1}(V)$ we mean the inverse of f applied to *elements* of V. The function f is said to be continuous at a point x if for each open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(U) \subseteq V$ (Munkres, 102). A continuous function is continuous at every point in the domain.

The definition above is equivalent to the ϵ-δ definition of continuity, as well as the limit definition ($\lim_{x \to c} f(x) = f(c)$) of continuity. However, as mentioned previously, both of these definitions require some notion of distance, whereas the topological definition is independent of distance. Continuity is useful because it tells us something about open sets of the domain based off of open sets from the codomain. It tells us that a function not only maps points to points, but that, in some sense, it sends neighboring points to neighboring points. If we look at continuous functions from some topological space X to the real numbers, we can learn about the structure of X based on open sets of $\mathbb{R}$, which we are now well acquainted with. In particular, locally constant functions provide us with information about the structure of their domain.

Example 1.12. (Locally constant functions are continuous) We define a locally constant function to be a function $f : X \to Y$, where X and Y are topological spaces, such that for any $x \in X$ there exists an open neighborhood U of x where $f(U) = \{y\}$ for some constant $y \in Y$. Not every open neighborhood is sent to the same constant. Constant functions are a special case of locally constant functions.

To show f is continuous, we let $V \subseteq Y$ be an open set. Next we consider the set $f^{-1}(V) = \{x \in X | f(x) \in V\}$ and show that it is open in X. If $f^{-1}(V) = \emptyset$, then it is open by definition. Otherwise, let $a \in f^{-1}(V)$. Since f is locally constant, there exists an open set U_a of X such that $f(x) = y'$ for all $x \in U_a$. Since $a \in U_a$ we have $f(a) = y'$, which implies $y' \in V$ by $a \in f^{-1}(V)$. It follows that for all $x \in U_a$ we have $f(x) = y' \in V$, and so $U_a \subseteq f^{-1}(V)$. By Definition 1.11, this means that f is continuous.

Example 1.12 tells us that locally constant functions from a topological space to $\mathbb{R}$ are indeed continuous, but it does not give us any particularly useful information regarding the domain. As it turns out, locally constant functions can tell us if our domain is in one piece or split into several pieces. First, we must understand what it means for a topological space to be "in one piece."

Definition 1.13. (Connected spaces) Let X be a topological space. We say X is connected if there are no pairs of disjoint open subsets of X whose union is X. That is, X is connected if there does not exist open sets $U, V \subseteq X$ such that $U \cap V = \emptyset$ and $U \cup V = X$. A set satisfying these conditions is said to be a *connected space* (Munkres, 146).

The opposite of a connected space is very easy to figure out based on Definition 1.13. A topological space X is not connected if there exist disjoint open sets $U, V \subseteq X$ such that $U \cup V = X$. Going back to locally constant functions, it turns out that we can determine whether or not the domain of a locally constant function is connected based on the values of the function.

Example 1.14. (Locally constant functions on non-connected sets can take multiple values) Consider a space X which is not connected. Let U_1 and U_2 be two open subsets of X whose union is X. Define a function $f : X \to Y$ by $f := \begin{cases} f(x) = y, \text{ where } y \text{ is a constant in Y} & x \in U_1 \\ f(x) = y' \text{ where } y' \text{ is a constant in Y} & x \in U_2 \end{cases}$. We see that f is locally constant, since for any $x \in X$, either $x \in U_1$ or $x \in U_2$ and $f(x)$ is constant on either open set.

Example 1.14 is only part of what we were hoping to prove. To show that the values of a locally constant function determine the connectedness of the function's domain, we have to show that there is a difference between locally constant functions on non-connected sets and locally constant functions on connected sets. Well, if locally constant functions on non-connected sets can take multiple values, it should follow that locally constant functions on a connected set are constant. That is, locally constant functions on a connected set take a single value. For this statement to be true, we need an additional property for the codomain of the function. It must be *Hausdorff*.

Proposition 1.15. *A locally constant function whose domain is a connected set and whose codomain is Hausdorff takes a single value.*

Proof. Let $f : X \to Y$ be a locally constant function on a connected set X, with the codomain, Y being Hausdorff. By *Hausdorff*, we mean a topological space satisfying an additional property: that if $y, y' \in Y$ with $y \neq y'$, there exist two open neighborhoods $V, V' \subset Y$ such that $y \in V$, $y' \in V'$, and $V \cap V' = \emptyset$. Intuitively, we can think of Hausdorff spaces as topological spaces whose points can be "separated." Importantly, if Y is Hausdorff, then $Y - \{y\}$ is open, as we now show. Let $a \in Y - \{y\}$. It follows that $a \neq y$, and since Y is Hausdorff, there exist disjoint open neighborhoods U_a and V_a of a and y respectively. Since $a \in U_a$, we have $U_a \subset Y - \{y\}$ and thus $Y - \{y\}$ is open. This fact will be the crux of our proof.

Since f is locally constant, there exists an open set U of X such that $f(U) = \{y\}$, where y is some constant in Y. Now consider the set $f^{-1}(Y - \{y\})$. As we have shown, since Y is Hausdorff, $Y - \{y\}$ is open. Since f is locally constant, it is continuous, and therefore $f^{-1}(Y - \{y\})$ is open in X. This implies we have an open cover $\mathcal{U} = \{U, f^{-1}(Y - \{y\})\}$. But note that we have $U \cap f^{-1}(Y - \{y\}) = \emptyset$, which will contradict the connectedness of X unless either $U = \emptyset$ or $f^{-1}(Y - \{y\}) = \emptyset$. The case where both sets are empty implies $X = \emptyset$, which would mean f is not a function. Similarly, if $U = \emptyset$, then f is not a locally constant function, as U is the open set that f is constant on. It follows then that $f^{-1}(Y - \{y\}) = \{x \in X | f(x) \in Y - \{y\}\} = \emptyset$, which implies there is no $x \in X$ that does not map to the constant y. Therefore, f is constant.

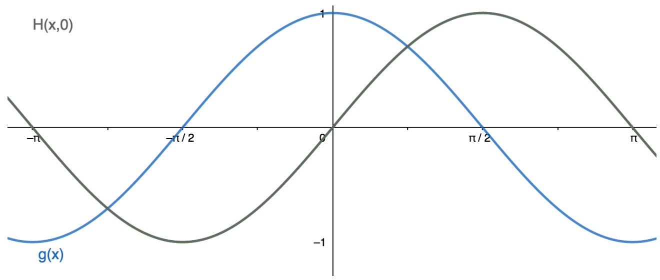
Example 1.14 and Proposition 1.15 explain why locally constant functions will be crucial to our study of Cech cohomology. Recall that Cech cohomology tells us about the structure of a topological space. Given an open cover $\{U_i\}_{i \in I}$ of a topological space X, if we look at the set of locally constant functions from each U_i to $\mathbb{R}$, we determine whether or not each U_i is a connected space. The 0-th cohomology group will tell us how disconnected a space is based on the U_i.

Additionally, locally constant functions allow us to discern whether or not a topological space is *contractible*. The idea of contractibility in our discussions can best be understood through homotopy.

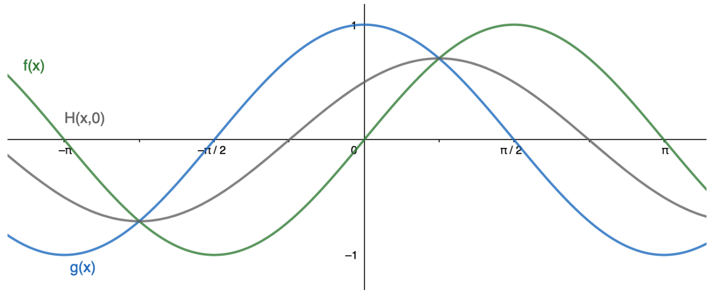
Definition 1.16 (Homotopy)**.** Let X and Y be topological spaces with continuous functions f and g between them. A *homotopy* between X and Y is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. If such a function exists, f is said to be *homotopic* to g, which we denote by $f \simeq g$. If g is a constant function, f is said to be *null-homotopic*. Intuitively, we can think of two functions as homotopic if one can be *continuously deformed* to another.

Example 1.17 ($sin(x)$ is homotopic to $cos(x)$)**.** Consider the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ where $f(x) = sin(x)$ and $g(x) = cos(x)$. Recall that $sin(x)$ and $cos(x)$ are both continuous. To show these functions are homotopic, we need a function $H : \mathbb{R} \times [0, 1] \to \mathbb{R}^2$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in \mathbb{R}$. If we let $H(x, t) = tcos(x) + (1 - t)sin(x)$, we have a continuous function where $H(x, 0) = 0 \cdot cos(x) + 1 \cdot sin(x) = 0 + sin(x) = f(x)$ and $H(x, 1) = 1 \cdot cos(x) + 0 \cdot sin(x) = cos(x) + 0 = cos(x) = g(x)$. Thus we say that $sin(x) \simeq cos(x)$. We can see from Figure 2 that this is the case.

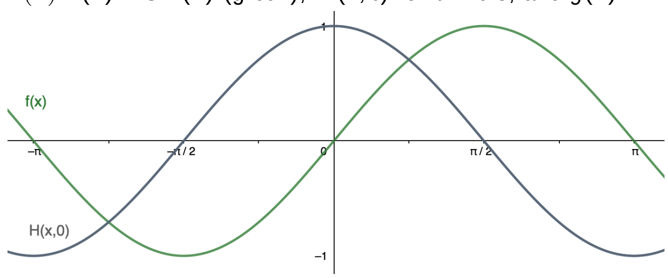
Example 1.18 (x^2 is homotopic to 3)**.** Consider $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ and $g(x) = 3$. Both of these functions are continuous, and we can define a continuous function $H : \mathbb{R} \times [0, 1] \to \mathbb{R}$ by $H(x, t) = (1 - t)x^2 + 3t$. It follows that $H(x, 0) = x^2 = f(x)$ and $H(x, 1) = 3 = g(x)$, and so we say that $x^2 \simeq 3$. That is, f is *null-homotopic*. We can also see this in Figure 3.



(A) $g(x) = cos(x)$ (blue) and $H(x;t)$ for $t = 0$ (gray).



(B) $f(x) = sin(x)$ (green), $H(x;t)$ for $t = 0.5$, and $g(x)$



(C) $f(x)$ and $H(x;t)$ for $t = 1$

FIGURE 2. The sine and cosine functions on the interval $[\quad;\quad]$ along with $H(x;t)$ for different values of t

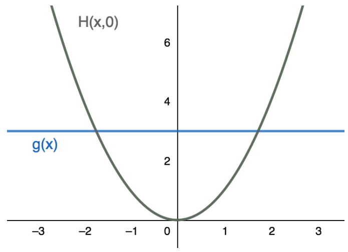
Note that in Examples 1.17 and 1.18, H used the pattern $H(x;t) := (1 \quad t) \quad f(x) + t \quad g(x)$. This pattern only holds when the codomain is a subset of $\mathbb{R}^n$ which is convex. The convex property is unique to Euclidean spaces, and as such we cannot use it unless we have a notion of distance in the codomain.

Now that we are familiar with homotopic functions, particularly null-homotopy, we can understand contractibility. There are multiple definitions for contractibility, but the one that follows naturally from null-homotopy is the easiest to understand.

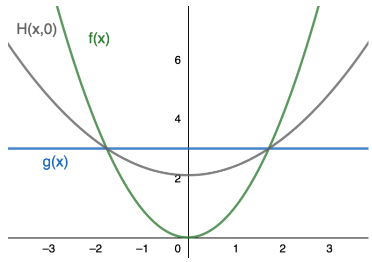
Definition 1.19 (Contractibile)**.** A topological space X is contractible if its identity map is null-homotopic. By identity map, we mean a function $id_X : X \to X$ such that $id_X(x) = x$ for all $x \in X$. Intuitively, we say that a topological space is contractible if it can be *continuously deformed* to a single point in the set.

Example 1.20 ($\mathbb{R}$ is contractible)**.** Consider the identity function on $\mathbb{R}$, denoted by $id_{\mathbb{R}}$. For a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = c$ for some constant $c \in \mathbb{R}$, we can define a function $H(x;t) := (1 \quad t)x + tc$. Since this is a continuous function satisfying $H(x;0) = id_{\mathbb{R}}$ and $H(x;1) = g(x)$, we have that the identity function $id_{\mathbb{R}}$ is homotopic to the constant function g. Since the identity function on $\mathbb{R}$ is null-homotopic, the set $\mathbb{R}$ is contractible.

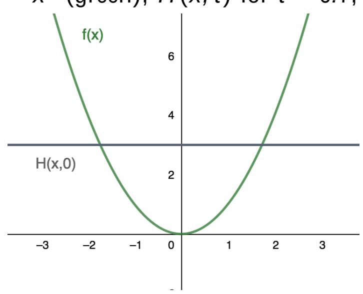
Example 1.21 (S^n is not contractible)**.** Consider a sphere in $\mathbb{R}^n$ with radius . Let x be a point on the surface of the sphere. Intuitively, then, we see that there can be no way to continuously deform all of the points on the n-sphere to x. If we "shrink" the radius, then the point x lies outside the sphere! To prove this formally, other methods of showing contractibility are required. However, these methods will be omitted, as they are tangential to our discussion.



(A) $g(x) = 3$ (blue) and $H(x, t)$ for $t = 0$ (gray).



(B) $f(x) = x^2$ (green), $H(x, t)$ for $t = 0.7$, and $g(x)$



(C) $f(x)$ and $H(x, t)$ for $t = 1$

FIGURE 3. The functions $f(x) = x^2$ and $g(x) = 3$ on the interval $[-3, 3]$ along with $H(x, t)$ for different values of t

Contractibility plays a pivotal role in Cech cohomology. Note that contractibility implies a set is connected. If a contractible space X were not connected, we could not "shrink" the disjoint open sets $U_1, U_2 \subset X$ to a single point without leaving X, and therefore X would not be contractible.

2. Algebra

Now that we know the necessary topology to understand Cech cohomology, we need to understand the algebra behind it. First we outline the basics of group theory, before moving on to homological algebra and defining cohomology groups. A rudimentary knowledge of linear algebra will be assumed, as it is only needed for computing the Cech cohomology. Basic properties of matrices and vector spaces, as well as the rank-nullity theorem, are all we need. Vector spaces will be briefly discussed, as they align perfectly with our discussion of R-modules.

Definition 2.1. (Groups and subgroups) Let G be a set. We say G is a group if there exists a *binary operation* on G, denoted by $* : G \times G \to G$ satisfying the following properties:

1. $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$ (associativity)
2. There exists an element g of G such that $a * g = a$ for all $a \in G$ (identity)
3. For every element a of G, there exists an element $a' \in G$ such that $a * a' = g$ (inverses)

If in addition to previous three properties we have that for all $a, b \in G$, $a * b = b * a$, then G is said to be a *commutative* or *abelian* group. A set $H \subset G$ is said to be a subgroup if the binary operation $*|_{H \times H}$ forms a group with elements of H.

Example 2.2. ($\mathbb{R}$ forms a group under addition) Consider the set $\mathbb{R}$ along with the binary operation, addition, denoted by $+$. Recall that $(\mathbb{R}, +)$ forms a group, a fact which rises from the axiomatic construction of the real numbers. We have that for any $a, b, c \in \mathbb{R}$, $a + (b + c) = (a + b) + c$. There is an element $0 \in \mathbb{R}$ such that for any $a \in \mathbb{R}$, $a + 0 = a$. Lastly, note that for any $a \in \mathbb{R}$ there exists an element $-a \in \mathbb{R}$ satisfying $a + (-a) = 0$. Thus, we see that $\mathbb{R}$ is a group under addition. Furthermore, we know that this is an abelian group, as $a + b = b + a$ for all $a, b \in \mathbb{R}$.

We could have shown just as easily that $\mathbb{R}^n$ is a group under addition. The proofs are essentially the same as Example 2.2, but the operation is defined on each component of an element of $\mathbb{R}^n$. It follows that since $\mathbb{R}$ is a group under the same operation(s), and since each component of $\mathbb{R}^n$ is in $\mathbb{R}$, all of the properties of abelian groups apply to $\mathbb{R}^n$.

We do not need real numbers to show a set is a group. For example, a set of functions can form a group. We just need to be a bit more careful with how we define our binary operation.

Example 2.3. (The set of functions G^S is a group) Given a set, S, and a group, G, we can define a set G^S as the set of functions $f : S \to G$. We wish to show that G^S forms a group. First we define a binary operation $\star : G^S \times G^S \to G^S$ by $(f \star g)(s) = *(f(s), g(s))$, where $f, g \in G^S$, $s \in S$ and $*$ is the binary operation on G.

Next, we wish to show that for any $f, g, h \in G^S$, we have $f \star (g \star h) = (f \star g) \star h$. Well, $f \star (g \star h) = *(f(s), *(g(s), h(s)))$ where $s \in S$. But since $f(s), g(s), h(s) \in G$ and G is a group, we have that $f \star (g \star h) = *(f(s), *(g(s), h(s))) = *(*(f(s), g(s)), h(s)) = (f \star g) \star h$.

Now we show that there exists an identity element of G^S. We denote our identity of G^S by the constant function $z(s) = e$, where e is the identity of G. Then for any $f \in G^S$, we have that $f \star z = *(f(s), z(s)) = *(f(s), e)$. Since G is a group, we have that $*(f(s), e) = f(s)$, which means $(f \star z)(s) = *(f(s), z(s)) = *(f(s), e) = f(s)$ for each $s \in S$ and so $f \star z = f$.

Lastly, we show that for any $f \in G^S$, there exists an element $g \in G^S$ such that $f \star g = z$. Such a function can be defined by $g(s) = f(s)'$, where $f(s)'$ is the inverse of the element $f(s)$ in the group G. Then $f \star g = *(f(s), g(s)) = *(f(s), f(s)') = e = z(s) = z$

Since G^S is associative, has an identity element, and inverse elements, G^S is a group under the binary operation $\star$.

Example 2.4. (The set of locally constant functions, R, forms an abelian group) Let X be a set. We define the set R to be the set of all locally constant functions from X to $\mathbb{R}$. The set R is just a special case of G^S from Example 2.3! The set of real numbers $\mathbb{R}$ is a group under addition. Since we are considering functions from a set to a group, we know that we can define a binary operation $\star$ on R by $f \star g = f(x) + g(x)$. To show that this forms a group, all we need to do is show it is closed. That is, we have to show $f \star g$ is also a locally constant function. Well, $f \star g = f(x) + g(x)$, where $f(x)$ and $g(x)$ are constants in $\mathbb{R}$. Their sum, $f(x) + g(x)$ will also be a constant, and so we see that $f \star g$ is also locally constant and $f \star g \in R$.

Note that we have an abelian group in this instance, since addition of real numbers $f(x)$ and $g(x)$ is commutative. Although it was easy to prove that R is a group, this fact will be immensely helpful in our discussion of Cech cohomology.

Particularly useful functions are ones that can "split up" the binary operations in the domain and codomain. Such functions are referred to as homomorphisms. However, the properties of a homomorphism vary depending on the structure of its domain and codomain. The most basic homomorphism we will consider is the group homomorphism, which have severeal interesting properties that apply to homomorphisms of other structures as well.

Definition 2.5. (Group homomorphisms) Let A and B be two groups with binary operations $*$ and $\star$ respectively. A group homomorphism is a function $f : A \to B$ satisfying $\star(f(a_1), f(a_2)) = f(*(a_1, a_2))$ for all $a_1, a_2 \in A$ and $f(a_1), f(a_2) \in B$

Proposition 2.6. *For a group homomorphism $f : A \to B$ we have that $ker(f) = \{a \in A | f(a) = e_B \in B\}$ is a subgroup of A and $im(f) = \{b \in B | f(a) = b$ for some $a \in A\}$ is a subgroup of B.*

Proof. Recall that a subset $H \subseteq G$ is a subgroup if it forms a group under the restriction of the binary operation on the group G. Let $*$ be the binary operation on A and $\star$ be the binary operation on B. Before we show that $ker(f)$ and $im(f)$ are subgroups, we first show that they maintain closure. Let $a, b \in ker(f)$. By definition, $ker(f) \subseteq A$, which means that $a, b \in A$, and therefore $*(a, b) \in A$. But $f(*(a, b)) = *(f(a), f(b))$, and since $a, b \in ker(f)$ we have that $*(f(a), f(b)) = *(e_B, e_B) = e_B \in B$. Thus, $*(a, b) \in ker(f)$ and we say that $ker(f)$ is closed under the binary operation $*$. Now let $x, y \in im(f)$. This means that there are elements $c, d \in A$ such that $f(c) = x$ and $f(d) = y$. Note that $im(f) \subseteq B$, so we have that $x, y \in B$ and $*(x, y) \in B$. Then $*(x, y) = *(f(c), f(d)) = f(*(c, d))$, where $*(c, d)$ is an element of A. Therefore, $*(x, y) \in im(f)$ and $im(f)$ is closed under the binary operation $*$.

Now we show that $ker(f)$ is a subgroup of A. Let $a, b, c \in ker(f)$. Note that these must also be elements of A, since $ker(f) \subseteq A$. Since A is a group under the operation $*$, we have that $*|_{ker(f) \times ker(f)}(a, *|_{ker(f) \times ker(f)}(b, c)) = *|_{ker(f) \times ker(f)}(*|_{ker(f) \times ker(f)}(a, b), c)$. Therefore, $ker(f)$ is associative under the restricted operation $*|_{ker(f) \times ker(f)}$.

The identity element of $ker(f)$ will be identical to the identity element of A, but of course we must show that this is true. Let e_A be the identity element of A and e_B be the identity of B. Well, $e_A = *(e_A, e_A)$ by definition. Then $f(e_A) = f(*(e_A, e_A)) = *(f(e_A), f(e_A))$. Since $f(e_A)$ is an element of the group B, there exists an element $f(e_A)^{-1}$ of B such that $*(f(e_A), f(e_A)^{-1}) = e_B$. Applying $f(e_A)^{-1}$ to both sides of $f(e_A) = *(f(e_A), f(e_A))$ we have that $*(*(f(e_A), f(e_A)), f(e_A)^{-1}) = *(f(e_A), f(e_A)^{-1}) \Rightarrow *(f(e_A), *(f(e_A), f(e_A)^{-1})) = *(f(e_A), f(e_A)^{-1}) \Rightarrow *(f(e_A), e_B) = e_B \Rightarrow f(e_A) = e_B$. Therefore, the identity element e_A of A is in $ker(f)$. But this does not tell us that e_A is the identity of $ker(f)$, so we must work a bit more. Let $a \in ker(f)$. Then $a \in A$ and $*|_{ker(f) \times ker(f)}(a, e_A) = *(a, e_A) = a$. Therefore, e_A is the identity element of $ker(f)$.

Lastly for $ker(f)$, we check for the existence of inverse elements. Let $a \in ker(f)$. Then we know that $a \in A$ and that there exists some $a' \in A$ such that $*(a, a') = e_A$. We now show that this same element a' must also be an element of $ker(f)$. Well, $e_B = f(e_A) = f(*(a, a')) = *(f(a), f(a'))$ and since $a \in ker(f)$ it follows that $*(f(a), f(a')) = *(e_A, f(a')) = f(*(e_A, a')) = f(a')$, so $a' \in ker(f)$. Finally we check that a' acts as an inverse under the restriction of $*$. $*|_{ker(f) \times ker(f)}(a, a') = *(a, a') = e_A$. Since $ker(f) \subseteq A$ is associative and has an identity element and inverses under the operation $*|_{ker(f) \times ker(f)}$, it is true that $ker(f)$ is a subgroup of A.

Now we show that $im(f)$ is a subgroup of B. We have already shown that $im(f)$ is closed under the operation $*$, so we can begin by showing $im(f)$ is associative. Let $x, y, z \in im(f)$. It follows that $x, y, z \in B$, and so we have that $*|_{im(f) \times im(f)}(x, *|_{im(f) \times im(f)}(y, z)) = *(x, *(y, z)) = *(*(x, y), z) = *|_{im(f) \times im(f)}(*|_{im(f) \times im(f)}(x, y), z)$.

Next we show $im(f)$ has an identity element. As one may suspect, the identity of $im(f)$ will be the same as the identity element of B. We have already shown that $e_B \in im(f)$, as we know that $f(e_A) = e_B$. Now we show that e_B does indeed act as an identity element. Let $x \in im(f)$. Then $x \in B$ and $*|_{im(f) \times im(f)}(x, e_B) = *(x, e_B) = x$ as desired.

Finally we show that there are inverses for each element of $im(f)$. We know that an element $x \in im(f)$ is an element of B and thus has an inverse $x' \in B$. To show that $x' \in im(f)$, we show that there exists some element $a' \in A$ such that $f(a') = x'$. Since $x \in im(f)$, there exists some $a \in A$ such that $f(a) = x$. Since $a \in A$, there also exists some element $a' \in A$ such that $*(a, a') = e_A$. Note that $x' = *(x', e_B) = *(x', f(e_A)) = *(x', f(*(a, a'))) = *(x', *(f(a), f(a'))) = *(*(x', f(a)), f(a')) = *(*(x', x), f(a')) = *(e_B, f(a')) = f(a')$. Since $x' = f(a')$ for some $a' \in A$, $x' \in im(f)$. But because $x' \in B$, it follows immediately that $*|_{im(f) \times im(f)}(x, x') = *(x, x') = e_B$. Since $im(f) \subseteq B$ is associative under $*$, and has an identity element and inverses, $im(f)$ is a subgroup of B.

Proposition 2.7. *For a group homomorphism $f : A \to B$, f is injective if and only if $ker(f) = \{e_A\}$ and f is surjective if and only if $im(f) = B$*

Proof. We first show that a group homomorphism f is injective provided $ker(f) = \{e_A\}$, where e_A is the identity of the group A. Suppose that for some $a, b \in A$ we have $f(a) = f(b)$. Since $f(a), f(b) \in B$ and B is a group, there exists a unique element $f(b)^{-1}$ such that $*(f(b), f(b)^{-1}) = e_B$, where $*$ is the binary operation on B and e_B is the identity element of B. Apply $f(b)^{-1}$ to both sides of $f(a) = f(b)$, we get $*(f(a), f(b)^{-1}) = *(f(b), f(b)^{-1}) \Rightarrow *(f(a), f(b)^{-1}) = e_B \Rightarrow f(*(a, b^{-1})) = e_B$, where $*(a, b^{-1}) \in A$ and b^{-1} is the inverse to the element $b \in A$. Thus $*(a, b^{-1}) \in ker(f)$. But since $ker(f) = \{e_A\}$, it follows that $*(a, b^{-1}) = e_A$. Applying b, the inverse of b^{-1}, to both sides yields $*(*(a, b^{-1}), b) = *(e_A, b) \Rightarrow *(a, *(b^{-1}, b)) = *(e_A, b) \Rightarrow *(a, e_A) = *(e_A, b) \Rightarrow a = b$. Therefore f is injective, just as we had hoped to prove.

Next we show that if f is injective, then $ker(f) = \{e_A\}$. Let $x \in ker(f)$. Then $f(x) = e_B$, and so $\{e_A\} \subseteq ker(f)$. But it is also true that $f(e_A) = e_B$. Consequently, $f(x) = f(e_A)$, and since f is injective this implies that $x = e_A$. Therefore the only unique element of $ker(f)$ is e_A. That is, $ker(f) \subseteq \{e_A\}$ and thus $ker(f) = \{e_A\}$.

Now we suppose that $im(f) = B$ and show that f must then be surjective. If $im(f) = B$, surjectivity follows by definition. Recall that $im(f) = \{b \in B | f(a) = b$ for some $a \in A\}$. Since $im(f) = B$, it follows that every element b in the codomain of f, there exists some element a in the domain of f such that $f(a) = b$. This is exactly what it means for a function to be surjective.

Finally, we check that if f is a surjective function, then $im(f) = B$. Once again, the conclusion follows immediately from the definition of the assumption. For a function g to be surjective, every element y in the codomain of g must have at least one element x in the domain of g such that $g(x) = y$. Thus, for any $b \in B$, we know there exists some $a \in A$ such that $f(a) = b$. In other words, $b \in im(f)$ and $B \subseteq im(f)$. We already know that $im(f) \subseteq B$, so we have $im(f) = B$.

Propositions 2.6 and 2.7 are important in understanding concepts of homological algebra. By use of Proposition 2.6, we will see that homology groups are in fact groups. Proposition 2.7 is crucial to proofs involving short exact sequences.

Similar to topological spaces, groups are not a very strict requirement for a set, and as such they will be used frequently. However, richer algebraic structures are also of use in the Cech cohomology. We now talk about rings, which will help us understand R-modules.

Definition 2.8. (Rings) Let R be an abelian group under addition. R is a ring if there is another binary operation, denoted by $\cdot$, satisfying associativity, and the following two properties: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$

Note that R is not required to be a group under $\cdot$, as neither an identity or inverses are necessary for a ring. **Needs reference.**

Example 2.9. ($\mathbb{R}$ is a ring) Consider the set $\mathbb{R}$, which we proved in Example 2.2 forms a group under addition. If we consider the operation $\cdot$ to be multiplication of real numbers, we can show that $\mathbb{R}$ is a ring. Let $a, b, c \in \mathbb{R}$. We have that multiplication of real numbers is associative, and that for any $a, b, c \in \mathbb{R}$, we have that $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c = b \cdot c$. Furthermore, $\mathbb{R}$ is an abelian group under $\cdot$, as we have the existence of an identity element, 1, the existence of inverses, and commutativity for multiplication of real numbers. All of these properties arise axiomatically from the definition of the set $\mathbb{R}$. Any ring satisfying the same properties is said to be a *field*.

Definition 2.10. (R-modules) Let G be a group under addition and R be a ring. A left R-module is formed by the set G and a binary operation $\cdot : R \times A \to A$ defined by $r \cdot a \in A$ and satisfying the following properties:

1. $r \cdot (a + b) = r \cdot a + r \cdot b$
2. $(r + s) \cdot a = r \cdot a + s \cdot a$
3. $r \cdot (s \cdot a) = (r \cdot s) \cdot a$

Note that the properties above hold only for a *left* R-module. A *right* R-module has exactly the same properties, but the elements of the group G are "hit" from the right by elements of the ring R. If the operation $\cdot$ is commutative, the module is said to be a *bi-module*. **Needs reference**

R-modules are used extensively in the Cech cohomology. In particular, they help us build a *Cech Complex*. The following are examples of some interesting R-modules

Example 2.11. (A binary operation $\cdot : \mathbb{Z} \times \mathbb{Q} \to \mathbb{Q}$ forms an R-module) Consider the set of rational numbers $\mathbb{Q}$. Note that this set is a group under addition, a structure which it passes to its superset, $\mathbb{R}$. Also note that the set of integers, $\mathbb{Z}$, forms a ring, as it is a group under addition, and satisfies basic associativity and distributivity under multiplication. Thus, we can form an R-module where $\mathbb{Z}$ is the ring. We show that the operation $\mathbb{Z} \times \mathbb{Q} \to \mathbb{Q}$ has the necessary properties for an R-module, starting with distributivity from the additive group. Since $\mathbb{Z} \subseteq \mathbb{Q}$, multiplication here is always between rational numbers. However, $\mathbb{Q}$ is a ring, as it also inherits multiplicative structure from $\mathbb{R}$. Thus, the binary operation $\cdot$ on the ring $\mathbb{Q}$ is essentially the same as the operation $\cdot : \mathbb{Z} \times \mathbb{Q} \to \mathbb{Q}$, and we will use them interchangeably. Then for any $r, s \in \mathbb{Z}$ and $a, b \in \mathbb{Q}$, we have that $r \cdot (a + b) = r \cdot a + r \cdot b$, since $r \in \mathbb{Q}$ and $\mathbb{Q}$ is a ring. Similarly, $(r + s) \cdot a = r \cdot a + s \cdot a$

because s is also an element of $\mathbb{Q}$. Lastly, we have $r \cdot (s \cdot a) = (r \cdot s) \cdot a$ because the ring operation $\cdot$ on $\mathbb{Q}$ must be associative. Therefore, the operation $\mathbb{Z} \times \mathbb{Q} \to \mathbb{Q}$ forms an R-module with the additive group $\mathbb{Q}$.

Most of what we did in Example 2.11 relied on the fact that $\mathbb{Z} \subseteq \mathbb{Q}$. As it turns out, any *subring* forms a module with its parent ring. By subring, we mean a subset of a ring which preserves the additive and multiplicative structures well enough to be considered a ring. For example, we could have shown just as easily that an operation $\varphi : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ forms an R-module with the additive group $\mathbb{R}$, since $\mathbb{Q}$ is a subring of $\mathbb{R}$.

Example 2.12. (Vector spaces are modules) Vector spaces are R-modules where the ring R is a field and the additive group is abelian. For our discussions, we consider the modules where the field is $\mathbb{R}$. The properties of a vector space V are as follows:

1. For all $u, v, w \in V$, $u + (v + w) = (u + v) + w$
2. There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$
3. For all $v \in V$, there exists an element $-v \in V$ such that $v + (-v) = 0$
4. For all $v, w \in V$ we have $v + w = w + v$
5. $1v = v$ for all $v \in V$, and where 1 denotes the identity element for multiplication in $\mathbb{R}$
6. For all $\alpha, \beta \in \mathbb{R}$ and $v \in V$, $\alpha(\beta v) = (\alpha\beta)v$
7. For all $\alpha \in \mathbb{R}$ and $v, w \in V$ we have $\alpha(u + v) = \alpha u + \alpha v$
8. For all $\alpha, \beta \in \mathbb{R}$ and $v \in V$ we have $(\alpha + \beta)v = \alpha v + \beta v$ **Probably needs a reference.**

The first four properties are satisfied by an abelian group V. Properties 6-8 are necessary for any R-module. Property 5 may not always be applicable for an R-module, since rings are not required to have identity elements for their secondary operation. However, if our ring is a field, then we are guaranteed a multiplicative identity.

Example 2.13. ($\mathbb{R} \times R$ is a vector space) Consider the set R of locally constant functions from a topological space X to the real numbers. We show that R is a vector space by showing it is an $\mathbb{R}$-module with a multiplicative identity. First, define addition by $f + g = f(x) + g(x)$. For vector spaces, we need scalar multiplication, which we can obtain naturally by defining $r \cdot f = r \cdot f(x)$ for any $r \in \mathbb{R}$, $f \in R$, and $x \in X$. Note that since $f(x) \in \mathbb{R}$, it follows that $r \cdot f(x) \in \mathbb{R}$, and therefore we have defined a map $\varphi : \mathbb{R} \times R \to R$, since there exists a locally constant function from X to $\mathbb{R}$ such that $g(x) = r \cdot f(x)$. As discussed in Examples 2.4 and 2.7, R is an abelian group under addition, and $\mathbb{R}$ is a ring. Next we check that φ satisfies the necessary properties for R-modules.

Let $r, s \in \mathbb{R}$ and $f, g \in R$. Then $r \cdot (f + g) = r \cdot (f(x) + g(x))$ for some $x \in X$. Since $f(x), g(x) \in \mathbb{R}$ we have that $r \cdot (f(x) + g(x)) = r \cdot f(x) + r \cdot g(x)$. Similarly, $(r + s) \cdot f = (r + s) \cdot f(x)$ for some $x \in X$, and since $f(x) \in \mathbb{R}$, we have $(r + s) \cdot f(x) = r \cdot f(x) + s \cdot f(x)$. Lastly, we see that $r \cdot (s \cdot f) = r \cdot (s \cdot f(x))$ where $x \in X$ and $f(x) \in \mathbb{R}$. Then $r \cdot (s \cdot f(x)) = (r \cdot s) \cdot f(x)$. Therefore, R is an $\mathbb{R}$-module.

Lastly, we show that this module has a multiplicative identity. Well, we know that for any $x \in X$ and $f \in R$ we have $f(x) = c$, where c is some constant in $\mathbb{R}$. Then for the multiplicative identity $1 \in \mathbb{R}$ and any $f \in R$ we have $1 \cdot f = 1 \cdot f(x) = f(x)$. Therefore, R is a vector space.

Examples 2.12 and 2.13 are very useful for defining the *Cech Complex*. Sets of the Cech Complex are direct products of sets of locally constant functions. It follows that this is a product of vector spaces, which is itself a vector space. The properties of vector spaces allow us to compute the Cech cohomology by hand. However, just as how we are thinking of vector spaces as $\mathbb{R}$-modules, we must rethink matrices (linear maps) as R-module homomorphisms.

Definition 2.14. (R-module homomorphisms) Let $f : A \to B$ be a function between two R-modules sharing the same ring. f is said to be an R-module homomorphism if for all $a, c \in A$ and $r \in R$: $f(a+c) = f(a)+f(c)$ and $f(r \cdot a) = r \cdot f(a)$ (Hungerford, 170).

Note that while we only showed Proposition 2.6 and Proposition 2.7 to be true for group homomorphisms, they can be extended to hold for homomorphisms of other structures as well. We could show, for example, that for a R-module homomorphism f, $ker(f)$ is a sub-module of its domain.

2.1. **Homological Algebra.** As its name suggests, Cech cohomology is ultimately a way of computing the *cohomology groups* for a topological space X. If we wish to understand Cech cohomology, we must first understand what homology groups are. Now that we know a bit of group theory, we can learn some homological algebra to familiarize ourselves with homology groups.

Definition 2.15. (Equivalence relations and equivalence classes) Let A, and B, be sets. A set R is a relation from A to B if $R \subseteq A \times B$. A relation R from A to itself is said to be an equivalence relation on A if the following properties hold:

1. For all $a \in A$ we have $(a, a) \in R$ (reflexive)
2. If $(a, b) \in R$ then $(b, a) \in R$ (symmetric)
3. If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ (transitive)

For an equivalence relation R on A, if $(a, b) \in R$, we write $a \sim b$ and say that a is equivalent to b under R. An equivalence class of an element $a \in A$ is the set of all elements of A equivalent to a under an equivalence relation R, denoted by the set $[a] = \{b \in A | b \sim a\}$ (Hungerford, 6).

Definition 2.16. (Normal subgroup) Let $(G, *)$ be a group. The *left coset* of an element $a \in G$ is given by $aG = \{*(a, g) \in G |$ for some $g \in G\}$. Similarly, the *right coset* of a is given by $Ga = \{*(g, a) \in G | for some g \in G\}$. For a subgroup H of G, H is said to be a normal subgroup if $hG = Gh$ for all $h \in H$ (Hungerford, 41).

Definition 2.17. (Quotient groups) Let H be a normal subgroup of $(G, *)$. Using an equivalence relation $g \sim g'$ if $gH = g'H$, we define the quotient group G/H, which is given the binary operation $*$. For any two elements $[g]$ and $[g']$ of G/H, we have that $g, g' \in G$ and $*([g], [g']) = [*(g, g')]$. (Hungerford, 42).

For our discussions, we only need to consider quotient groups under addition or multiplication. Interestingly, there is another method to check equivalency for additive abelian groups. We say that for an additive abelian group G/H with classes $[g]$ and $[g']$, we have that $g \sim g'$ if $g - g' \in H$. For a multiplicative group, we have $g \sim g'$ if $(g')^{-1} \cdot g \in H$. We will use these methods of equivalency instead of the formal method in Definition 2.17.

Example 2.18. ($\mathbb{Z}/5\mathbb{Z}$ is a quotient group) Consider the set $\mathbb{Z}/5\mathbb{Z}$ under multiplication. We can show that this is a group, but first we explore what elements are in this set. By $5\mathbb{Z}$, we mean the set $\{\dots, -5, 0, 5, 10, \dots\}$, all of the multiples of 5. Two classes $[a], [a'] \in \mathbb{Z}/5\mathbb{Z}$ are equivalent if $a - a' \in 5\mathbb{Z}$. For example, we have that $[1] = [6]$, since $1 - 6 = -5 \in 5\mathbb{Z}$. Intuitively, then, we can think of the set $\mathbb{Z}/5\mathbb{Z}$ as all of the possible remainders of division by 5. That is, $\mathbb{Z}/5\mathbb{Z} = \{[0], [1], [2], [3], [4]\}$. We show that this is a group under multiplication through the use of the following table:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

From the table we can observe the following:

- $\mathbb{Z}/5\mathbb{Z}$ is closed
- There exists a multiplicative identity, $[1] \in \mathbb{Z}/5\mathbb{Z}$
- For each element $[a]$ of $\mathbb{Z}/5\mathbb{Z}$, there exists an inverse element $[a^{-1}] \in \mathbb{Z}/5\mathbb{Z}$ such that $[a] \cdot [a^{-1}] = [1]$
- The set $\mathbb{Z}/5\mathbb{Z}$ is commutative under multiplication

All we need to check for now is associativity. Let $[a], [b], [c] \in \mathbb{Z}/5\mathbb{Z}$. Then $[a] \cdot ([b] \cdot [c]) = [a \cdot (b \cdot c)] = [(a \cdot b) \cdot c] = ([a] \cdot [b]) \cdot [c]$. Therefore, $\mathbb{Z}/5\mathbb{Z}$ is an abelian group.

So far, we have talked about algebraic structures and their homomorphisms somewhat exclusively. Homological algebra frequently requires one to think about the two ideas in conjunction. We can tie algebraic structures and their homomorphisms together the following definition.

Definition 2.19. (Chain complex) A chain complex is a sequence of groups and their homomorphisms, denoted by

$$(C_\bullet, d_\bullet) := \cdots \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \cdots$$

and satisfying $d^{p+1} \; d^p = 0$. Informally, we can denote the property $d^{p+1} \; d^p = 0$ by "$d^2 = 0$" for the sake of brevity. To most, the definition here applies to *cochain complexes* instead of chain complexes, but for *cohomology*, we only need to consider cochain complexes. We will use "chain complex" and "homology" interchangeably with "cochain complex" and "cohomology" to reduce the number of times we use "co's."

Example 2.20. (The sequence $0 \xrightarrow{\ ^{1}} \mathsf{R} \xrightarrow{\ ^{0}} \mathsf{R}^3 \xrightarrow{\ ^{1}} \mathsf{R}^2 \xrightarrow{\ ^{2}} 0$ forms a chain complex) Consider the sequence $0 \xrightarrow{\ ^{1}} \mathsf{R} \xrightarrow{\ ^{0}} \mathsf{R}^3 \xrightarrow{\ ^{1}} \mathsf{R}^2 \xrightarrow{\ ^{2}} 0$. We can show that this is a chain complex by constructing functions ^{0} and ^{1} satisfying the requirement $d^2 = 0$. Note that we do not need to define ^{1} or ^{2}, since the domain of ^{1} is 0 and ^{2} will map everything to 0 anyway. We can check that $d^2 = 0$ without defining those maps. It will be assumed that R^3 and R^2 are groups under addition, as this is a fact we can verify for any R^n in a proof that is nearly identical to Example 2.2.

Now we define a map $^{0} : \mathsf{R} \to \mathsf{R}^3$ and $^{1} : \mathsf{R}^3 \to \mathsf{R}^2$ such that $^{1} \; ^{0} = 0$. Define $^{0}(x) = (x, 0, 0) \in \mathsf{R}^3$ and $^{1}(x, y, z) = (0, y) \in \mathsf{R}^2$. It follows then that $^{1} \; ^{0} = \; ^{1}(\; ^{0}(x)) = \; ^{1}(x, 0, 0) = (0, 0) = 0 \in \mathsf{R}^2$. Thus, we see that $^{2} = 0$. Next we check that ^{0} and ^{1} are both group homomorphisms. For any $x_1, x_2 \in \mathsf{R}$ we have $^{0}(x_1+x_2) = (x_1+x_2, 0, 0)$ and $^{0}(x_1) + \; ^{0}(x_2) = (x_1, 0, 0) + (x_2, 0, 0) = (x_1+x_2, 0+0, 0+0) = (x_1+x_2, 0, 0)$, which of course means $^{0}(x_1 + x_2) = \; ^{0}(x_1) + \; ^{0}(x_2)$. Checking that ^{1} is a group homomorphism is just as simple. For any $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathsf{R}^3$, we have $^{1}((x_1, y_1, z_1) + (x_2, y_2, z_2)) = \; ^{1}((x_1 + x_2, y_1 + y_2, z_1 + z_2) = (0, y_1+y_2)$. But we also have that $^{1}((x_1, y_1, z_1)) + \; ^{1}((x_2, y_2, z_2)) = (0, y_1) + (0, y_2) = (0+0, y_1+y_2) = (0 + y_2)$, and so we say $^{1}((x_1, y_1, z_1) + (x_2, y_2, z_2)) = \; ^{1}((x_1, y_1, z_1)) + \; ^{1}((x_2, y_2, z_2))$.

Note that ^{1} and ^{2} are also group homomorphisms, which we can check without defining them. The map ^{1} must necessarily be a group homomorphism, since its domain and codomain consist only of 0. As for ^{2}, we have for any $(x_1, y_1), (x_2, y_2) \in \mathsf{R}^2$ that $^{2}((x_1, y_1) + (x_2, y_2)) = \; ^{2}((x_1 + x_2, y_1 + y_2) = 0$ and $^{2}((x_1, y_1)) + \; ^{2}((x_2, y_2)) = 0 + 0 = 0$, which implies $^{2}((x_1, y_1) + (x_2, y_2)) = \; ^{2}((x_1, y_1)) + \; ^{2}((x_2, y_2))$.

Now we check that $d^2 = 0$ Since ^{0} is a group homomorphism, we can see that $^{0} \; ^{1} = \; ^{0}(0)$, which maps to the 0 of the codomain of ^{0}. Similarly, we see that $^{2} \; ^{1}$ must map to 0, since $^{2} : \mathsf{R}^2 \to 0$.

Therefore, the sequence $0 \xrightarrow{\ ^{1}} \mathsf{R} \xrightarrow{\ ^{0}} \mathsf{R}^3 \xrightarrow{\ ^{1}} \mathsf{R}^2 \xrightarrow{\ ^{2}} 0$ is a chain complex.

Definition 2.21. (Chain map) A chain map is a collection of homomorphisms, f, between two chain complexes (A, g) and (B, h) satisfying the commutative diagram. That is, for any $f^p, f^{p+1} \in f$ we have $f^{p+1} \; g^p = h^p \; f^p$.

$$
\begin{array}{ccccccccc}
\dots & \xrightarrow{g^{p-2}} & A^{p-1} & \xrightarrow{g^{p-1}} & A^p & \xrightarrow{g^p} & A^{p+1} & \xrightarrow{g^{p+1}} & \dots \\
 & & \downarrow{\scriptstyle f^{p-1}} & & \downarrow{\scriptstyle f^p} & & \downarrow{\scriptstyle f^{p+1}} & & \\
\dots & \xrightarrow{h^{p-2}} & B^{p-1} & \xrightarrow{h^{p-1}} & B^p & \xrightarrow{h^p} & B^{p+1} & \xrightarrow{h^{p+1}} & \dots
\end{array}
$$

Needs reference.

Homology groups are based off of chain complexes. The aptly named Cech Complex is itself a chain complex. From prior comments regarding the Cech Complex, we see that the sets of the Cech Complex, generated by locally constant functions of an open cover, form a sequence of R-modules. The homomorphisms, as well as the Cech Complex itself will be formally defined in Section 5.

Chain maps are not useful in computing the Cech cohomology by hand, but are immensely helpful to those seeking to make an algorithm for the Cech cohomology. This discussion will be saved for Section 5.2, where we will have the necessary definitions at our disposal. For now, we familiarize ourselves with exact sequences before moving on to a definition of homology groups.

Definition 2.22. (Exact, short, and long sequences) An exact sequence is a chain complex $(C, \;)$ where $im(\; ^{p-1}) = ker(\; ^{p})$. If an exact sequence contains only three nontrivial groups, that is, if the sequence is in the form $0 \to A \to B \to C \to 0$ the sequence is said to be short exact. An exact sequence with more than three consecutive nontrivial groups is said to be a long exact sequence. **(Needs a reference.)**

Lemma 2.23. *If the sequence* $0 \xrightarrow{d^{-1}} A \xrightarrow{d^0} B \xrightarrow{d^1} C \xrightarrow{d^2} 0$ *is exact, then* d^0 *is injective and* d^1 *is surjective.*

Proof. Suppose the sequence $0 \xrightarrow{d^{-1}} A \xrightarrow{d^0} B \xrightarrow{d^1} C \xrightarrow{d^2}$ is exact. That is, suppose $im(d^{p-1}) = ker(d^p)$ for each d^p in the sequence. It follows, then, that $im(d^{-1}) = ker(d^0)$. But since $d^{-1} : 0 \to A$, we know that $im(d^{-1}) = 0$. Since $im(d^{-1}) = ker(d^0)$ via our assumption, we have that $ker(d^0) = 0$ and thus d^0 is injective by Proposition 2.7. Similarly, we have that $ker(d^2) = C$, since $d^2 : C \to 0$. Because $ker(d^2) = im(d^1)$, we have $im(d^1) = C$, which is the codomain of d^1. Therefore, by Proposition 2.7, d^1 is surjective, just as we set out to prove.

Definition 2.24. (Cohomology groups) Let $(C^\bullet, d^\bullet)$ be a chain complex of abelian groups. We define the cohomology group $H^\bullet(C^\bullet)$ by $H^p(C^\bullet) := ker(d^p)/im(d^{p-1})$ **(Needs a reference.)**

The following two theorems are useful for creating an algorithm to "track" homology groups of different chain complexes. As we will discuss in Section 5.2, such algorithms allow us to compute the *persistent homology* of a topological space. The results of these theorems are best understood after defining Cech cohomology formally, but we prove them in this section to familiarize ourselves with homology groups.

Theorem 2.25. *A chain map $f^\bullet : (C^\bullet, g^\bullet) \to (D^\bullet, h^\bullet)$ induces a map on the homology groups $H^\bullet(C^\bullet) \to H^\bullet(D^\bullet)$*

Proof. Really, this proof relies on the following lemma: If $f : G \to H$ is a group homomorphism and $A \subseteq G$ and $B \subseteq H$ are subgroups such that $f(A) \subseteq B$, then f induces a group homomorphism $\phi : G/A \to H/B$ given by $\phi([g]) = [f(g)]$. First we will prove the lemma, then show that it applies to the theorem.

We start by showing that ϕ is well-defined. Suppose $[g] = [g']$. Then $g - g' \in A$. Well, $\phi([g]) = [f(g)]$ and $\phi([g']) = [f(g')]$. But since f is a group homomorphism, we have that $f(g - g') = f(g) - f(g')$ is an element of B, and thus $[f(g)] = [f(g')] \Rightarrow \phi([g]) = \phi([g'])$.

Next we show ϕ is a group homomorphism. $\phi([g] + [g']) = \phi([g + g']) = [f(g + g')] = [f(g) + f(g')]$ since f is a group homomorphism. It follows that $\phi([g] + [g']) = [f(g) + f(g')] = [f(g)] + [f(g')] = \phi([g]) + \phi([g'])$. Thus we have proven the lemma.

We will rely heavily on the following diagram for the next part of this proof.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{g^{p-2}} & C^{p-1} & \xrightarrow{g^{p-1}} & C^{p} & \xrightarrow{g^{p}} & C^{p+1} & \xrightarrow{g^{p+1}} & \cdots \\
 & & \downarrow{\scriptstyle f^{p-1}} & & \downarrow{\scriptstyle f^{p}} & & \downarrow{\scriptstyle f^{p+1}} & & \\
\cdots & \xrightarrow{h^{p-2}} & D^{p-1} & \xrightarrow{h^{p-1}} & D^{p} & \xrightarrow{h^{p}} & D^{p+1} & \xrightarrow{h^{p+1}} & \cdots
\end{array}$$

To show that the lemma is applicable to the theorem above, we need to show that we have a map $f^p : ker(g^p) \to ker(h^p)$ and that $im(g^{p-1}) \subseteq ker(g^p)$, $im(h^{p-1}) \subseteq ker(h^p)$, and $f^p(im(g^{p-1}) \subseteq im(h^{p-1})$.

Let $x \in ker(g^p)$. We have then that $g^p(x) = 0 \in C^{p+1}$. It follows that $f^{p+1} \circ g^p(x) = 0 \in D^{p+1}$, and by the commutative diagram we have $h^p \circ f^p(x) = 0 \in D^{p+1}$, which implies $f^p(x) \in ker(h^p)$, yielding the desired map.

Next, let $a \in im(g^{p-1})$. This means $a = g^{p-1}(b)$ for some $b \in C^{p-1}$. Thus, $g^p(a) = g^p \circ g^{p-1}(b) = 0 \in C^{p+1}$ because $(C^\bullet, g^\bullet)$ is a chain complex. Therefore $g^{p-1}(b) = a \in ker(g^p)$ and $im(g^{p-1}) \subseteq ker(g^p)$. Recall that $ker(g^p)$ is a subgroup of the domain C^p, and that $im(g^{p-1})$ is a subgroup of the codomain C^p, and so we have $im(g^{p-1})$ is a subgroup of $ker(g^p)$.

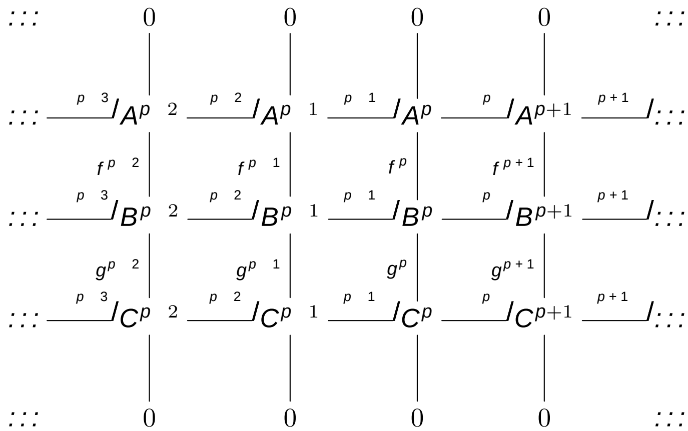
Now let $x \in im(h^{p-1})$. Then $x = h^{p-1}(y)$ for some $y \in D^{p-1}$. Since $(D^\bullet, h^\bullet)$ is a chain complex, the element x belongs to $ker(h^p)$, as $h^p \circ h^{p-1}(y) = h^p(x) = 0 \in D^{p+1}$. Thus $im(h^{p-1}) \subseteq ker(h^p)$ and $im(h^{p-1})$ is a subgroup of $ker(h^p)$.

Lastly, let $a \in im(g^{p-1})$. It follows that $a = g^{p-1}(b)$ for some $b \in C^{p-1}$. Then $f^p(a) = f^p \circ g^{p-1}(b) = k$ is an element of D^p. Since we are working in a commutative diagram, it follows that $h^{p-1} \circ f^{p-1}(b) = k$, which implies $k = h^{p-1}(j)$ for some $j = f^{p-1}(b) \in D^{p-1}$. Thus $f^p(a) = k \in im(h^{p-1})$ and $f(im(g^{p-1}) \subseteq im(h^{p-1})$.

Therefore, we can use the lemma above to define a map $\phi^p : ker(g^p)/im(g^{p-1}) \to ker(h^p)/im(h^{p-1})$ by $\phi^p([x]) = [f^p(x)]$. Thus we have a map $\phi^\bullet : H^\bullet(C^\bullet) \to H^\bullet(D^\bullet)$ induced by the chain map $f^\bullet$.

Theorem 2.26. *(A short exact sequence of chain complexes implies a long exact sequence of cohomology groups).*

Proof. The following diagram will be referenced extensively for much of this proof.



We are given a sequence $0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$ where $f^\bullet$ and $g^\bullet$ are chain maps between the chain complexes $(A^\bullet, d_A), (B^\bullet, d_B),$ and $(C^\bullet, d_C)$ satisfying a short exact sequence $0 \to A^p \xrightarrow{f^p} B^p \xrightarrow{g^p} C^p \to 0$ for each p. What we want to show is that these conditions induce a long exact sequence of cohomology groups in the form $\cdots \to H^{p-1}(C^\bullet) \xrightarrow{\delta^{p-1}} H^p(A^\bullet) \xrightarrow{\phi_A^p} H^p(B^\bullet) \xrightarrow{\phi_B^p} H^p(C^\bullet) \xrightarrow{\delta^p} H^{p+1}(A^\bullet) \to \cdots$

Recall from Theorem 2.24 that the maps ϕ_A and ϕ_B are defined by $\phi_A^p([x]) = [f^p(x)]$ and $\phi_B^p([y]) = [g^p(y)]$ for some $[x] \in H^p(A^\bullet)$ and $[y] \in H^p(B^\bullet)$. Since we have already proved that these maps are well-defined group homomorphisms, we can ignore them until we need to show that the sequence of homology groups is exact. Instead we direct our attention to the collection of maps δ, where $\delta^p : H^{p-1}(C^\bullet) \to H^p(A^\bullet)$. Before we define such a map, we first show that there is a map $ker(d_C^{p-1}) \to ker(d_A^p)$.

Let $c \in ker(d_C^{p-1})$. Then c is an element of C^{p-1} such that $d_C^{p-1}(c) = 0 \in C^p$. Since $c \in C^{p-1}$ and g^{p-1} is surjective by Lemma 2.22, we have that $c \in im(g^{p-1})$. Thus there exists some $y \in B^{p-1}$ such that $g^{p-1}(y) = c$. Since this is a commutative diagram, and we have that $d_C^{p-1} \circ g^{p-1}(y) = 0 \in C^p$, we have that $g^p \circ d_B^{p-1}(y) = 0 \in C^p$, which implies $d_B^{p-1}(y) \in ker(g^p)$. It follows then that $d_B^{p-1} \in im(f^p)$, since in an exact sequence we have $ker(g^p) = im(f^p)$. Thus there exists some element $x \in A^p$ such that $f^p(x) = d_B^{p-1}(y)$. Note that x is unique, since the function f^p is injective. Now we wish to show that $x \in ker(d_A^p)$. Well, $d_B^p \circ d_B^{p-1}(y) = 0 \in B^{p+1}$ because we have a chain complex $(B^\bullet, d_B)$. But recall that $f^p(x) = d_B^{p-1}(y)$, so we have that $d_B^p \circ f^p(x) = 0 \in B^{p+1}$. Since this is a commutative diagram, $f^{p+1} \circ d_A^p(x) = 0 \in B^{p+1}$ and $d_A^p(x) \in ker(f^{p+1})$. The function f^{p+1} is injective, so it follows that $ker(f^{p+1}) = \{0\}$ and $d_A^p(x) = 0$. Therefore, $x \in ker(d_A^p)$, meaning there exists a map $ker(d_C^{p-1}) \to ker(d_A^p)$. Importantly, this means that there exists a class $[x] \in H^p(A^p)$. This allows us to define a map $\delta^{p-1} : H^{p-1}(C^\bullet) \to H^p(A^\bullet)$ by $\delta^{p-1}([c]) = [x]$. However, recall that when we derived the element $y \in B^{p-1}$, we were not guaranteed that this element was unique. To check that δ^{p-1} is well-defined, we must check that $[x]$ is independent of the choice of y, and then show that if $[c] = [c'] \in C^{p-1}$ then $\delta^{p-1}([c] = \delta^{p-1}([c'])$.

Suppose there exist two elements $y, y' \in B^{p-1}$ such that $g^{p-1}(y) = g^{p-1}(y') = c \in C^{p-1}$, where c was the element we assumed belonged to $ker(d_C^{p-1})$. It follows then, since we are working in chain complexes of abelian groups, that $g^{p-1}(y) - g^{p-1}(y') = 0 \Rightarrow g^{p-1}(y - y') = 0 \in C^{p-1} \Rightarrow y, y' \in ker(g^{p-1})$. It follows by the property of a short exact sequence that $ker(g^{p-1}) = im(f^{p-1})$. Then there exists a unique element $a \in A^{p-1}$ such that $f^{p-1}(a) = y - y' \in B^{p-1}$. Note that $d_C^{p-1} \circ g^{p-1}(y) = d_C^{p-1} \circ g^{p-1}(y') = 0 \in C^p$, and since we have a commutative diagram this implies $g^p \circ d_B^{p-1}(y) = g^p \circ d_B^{p-1}(y') = 0 \in C^p$. Then $d_B^{p-1}(y), d_B^{p-1}(y') \in ker(g^p) \Rightarrow d_B^{p-1}(y), d_B^{p-1})y') \in im(f^p)$. Then there exist unique elements $x, x' \in A^p$ such that $f^p(x) = d_B^{p-1}(y)$ and $f^p(x') = d_B^{p-1}(y')$. But note that $g^p \circ d_B^{p-1}(y - y') = g^p(d_B^{p-1}(y) - d_B^{p-1}(y')) = g^p \circ d_B^{p-1}(y) - g^p \circ d_B^{p-1}(y') = 0 \in C^p$ and $d_B^{p-1}(y - y') \in ker(g^p)$. We see then that $d_B^{p-1}(y - y') \in im(f^p)$. We already know which element of A^p maps to $d_B^{p-1}(y - y')$, since x, x' are unique and for the element $x - x' \in A^p$ we have $f^p(x - x') = f^p(x) - f^p(x') = d_B^{p-1}(y) - d_B^{p-1}(y') = d_B^{p-1}(y - y')$. Recall that $f^{p-1}(a) = y - y'$. Thus, $d_B^{p-1} \circ f^{p-1}(a) = d_B^{p-1}(y - y')$. As we have just shown, $f^p(x - x') = d_B^{p-1}$, and since this is a commutative diagram, $d_B^{p-1} \circ f^{p-1}(a) = f^p \circ d_A^{p-1}(a) = d_B^{p-1}(y - y')$. Because f^p is injective, it follows that since $f^p \circ d_A^{p-1}(a) = f^p(x - x') = d_B^{p-1}(y - y')$, we have $d_A^{p-1}(a) = x - x'$. This means that $x - x' \in im(d_A^{p-1})$. Recall that x and x' both have classes $[x], [x'] \in H^p(A^p)$. But since $x - x' \in im(d_A^{p-1})$, by definition of the quotient group $H^p(A^p) = ker(\partial^p)/im(\partial^{p-1})$ that $[x] = [x']$. Therefore, the class $[x]$ is independent of our choice of y.

Next, we show that δ^{p-1} is well-defined. Suppose $[c] = [c'] \in H^{p-1}(C^{p-1})$. Then $c - c' \in im(\partial^{p-2})$. Well, $c - c'$ must be equal to some element of $im(\partial^{p-2})$, so $c - c' = \partial^{p-2}(z)$ for some $z \in C^{p-2}$. This implies $c = \partial^{p-2}(z) + c'$. Since g^{p-2} is surjective, there exists some $q \in B^{p-2}$ such that $g^{p-2}(q) = z$. Similarly, since g^{p-1} is surjective, there exist elements $p, y \in B^{p-1}$ such that $g^{p-1}(p) = \partial^{p-2}(z)$ and $g^{p-1}(y) = c$. This implies that $g^{p-1}(p + y) = g^{p-1}(p) + g^{p-1}(y) = \partial^{p-2}(z) + c = c'$. Recall that $[c], [c'] \in H^{p-1}(C^{p-1})$ implies $c, c' \in ker(\partial^{p-1})$. Then $\partial^{p-1}(c) = \partial^{p-1}(c') = 0 \in C^p$. We have then that $\partial^{p-1} \circ g^{p-1}(y) = 0 \in C^p$ and $\partial^{p-1} \circ g^{p-1}(p + y) = 0 \in C^p$. Because this is a commutative diagram, it follows that $g^p \circ \partial^{p-1}(y) = 0$ and $g^p \circ \partial^{p-1}(p + y) = 0$. Note that $\partial^{p-1}(p + y) = \partial^{p-1}(p) + \partial^{p-1}(y)$, but since $\partial^{p-2} \circ g^{p-2}(q) = \partial^{p-2}$ and $g^{p-1}(p) = \partial^{p-2}$, by the commutative diagram we have that $g^{p-1} \circ \partial^{p-2}(q) = \partial^{y-2}(z)$. Thus, $\partial^{p-2}(q) = p$, and $\partial^{p-1}(p) = \partial^{p-1} \circ \partial^{p-2}(q) = 0 \in B^p$. Then $\partial^{p-1}(p + y) = \partial^{p-1}(p) + \partial^{p-1}(y) = \partial^{p-1}(y)$. We see then that the resulting x which maps to $\partial^{p-1}(y)$ and x' which maps to $\partial^{p-1}(y+p)$ will be the same, which implies $[x] = [x'] \Rightarrow \delta^{p-1}([c]) = \delta^{p-1}([c'])$. Therefore, our map δ^{p-1} is well defined.

Note that since $H^{p-1}(C^{p-1})$ and $H^p(A^p)$ are groups, we must show that δ^{p-1} is a group homomorphism. Well, $\delta^{p-1}([c]) + \delta^{p-1}([c']) = [x] + [x'] = [x + x']$. What we really want to show is that $\delta^{p-1}([c] + [c']) = \delta^{p-1}([c + c']) = [x + x']$. Well, $[c + c'] \in H^{p-1}(C^{p-1})$ implies $c + c' \in ker(\partial^{p-1})$. Thus, $\partial^{p-1}(c + c') = 0 \in C^p$. Since g^{p-1} is surjective, there exist elements $y, y' \in B^{p-1}$ such that $g^{p-1}(y) = c$ and $g^{p-1}(y') = c'$. It follows naturally that $c + c' = g^{p-1}(y) + g^{p-1}(y') = g^{p-1}(y + y')$. Then $\partial^{p-1} \circ g^{p-1}(y + y') = 0 \in C^p$, and since this is a commutative diagram $g^p \circ \partial^{p-1}(y + y') = 0 \in C^p$ Then $\partial^{p-1}(y + y') \in ker(g^p)$. Note that since $c, c' \in ker(\partial^{p-1})$ and $g^{p-1}(y) = c$ and $g^{p-1}(y') = c'$ it is true for the same reasons that $\partial^{p-1}(y), \partial^{p-1}(y') \in ker(g^p)$. Then $\partial^{p-1}(y + y'), \partial^{p-1}(y), \partial^{p-1}(y') \in im(f^p)$ via the exact sequence. Since f is injective, this means there are unique elements $x, x' \in A^p$ such that $f^p(x) = \partial^{p-1}(y)$ and $f^p(x') = \partial^{p-1}(y')$. This means that we also have $f^p(x + x') = f^p(x) + f^p(x') = \partial^{p-1}(y) + \partial^{p-1}(y') = \partial^{p-1}(y + y')$. Since $\partial^p \circ \partial^{p-1} = 0 \in B^p$, it follows that $\partial^p \circ f^p(x + x') = \partial^p \circ \partial^{p-1}(y + y') = 0 \in B^p$. Using the commutative property of this diagram yields $f^{p+1} \circ \partial^p(x + x') = 0 \in B^p$, which means $\partial^p(x + x') \in ker(f^{p+1})$. Because f^{p+1} is injective this means $\partial^p(x + x') = 0 \in A^{p+1}$ and $x + x' \in ker(\partial^p)$. Thus $x + x'$ has a class $[x + x'] \in H^p(A^p)$, which we derived from $[c + c']$. Therefore $\delta^{p-1}([c + c']) = [x + x'] = [x] + [x'] = \delta^{p-1}([c]) + \delta^{p-1}[c']$ and δ^{p-1} is a group homomorphism.

Lastly, we show that the sequence $\ldots \to H^{p-1}(C) \xrightarrow{\delta^{p-1}} H^p(A) \xrightarrow{\varphi^p_A} H^p(B) \xrightarrow{\varphi^p_B} H^p(C) \xrightarrow{\delta^p} H^{p+1}(A) \to \ldots$ is exact. This requires us to show that $ker(\varphi^p_A) = im(\delta^{p-1})$, $ker(\varphi^p_B) = im(\varphi^p_A)$, and $ker(\delta^p) = im(\varphi^p_B)$.

Let $[a] \in im(\delta^{p-1})$. Then there exists a class $[c] \in H^{p-1}(C^{p-1})$ such that $\delta^{p-1}([c]) = [a] \in H^p(A^p)$. We also have that $\varphi^p_A([a]) = [f^p(a)]$. Recall that we have shown that $f^p(a)$ maps to some element $\partial^{p-1}(y) \in B^p$, where y is an element of B^{p-1} such that $g^{p-1}(y) = c$. It follows that $\partial^{p-1}(y) = f^p(a)$ and $f^p(a) \in im(\partial^{p-1})$. Since f^p and ∂^p are homomorphisms, for the identity element $0 \in A^p$ we have $\partial^p \circ f^p(0) = \partial^p(0) = 0 \in B^{p+1}$, and $f^p(0) \in ker(\partial^p)$. Therefore $f^p(0) = 0 \in B^p$ has a class $[0] \in H^p(B^p)$. But note that $f^p(a) - 0 = f^p(a) \in im(\partial^{p-1})$, so we have that $[0] = [f^p(a)] = \varphi^p_A([a])$. Then $[a] \in ker(\varphi^p_A)$ and $im(\delta^{p-1}) \subseteq ker(\varphi^p_A)$. Now we show equality. Let $[x] \in ker(\varphi^p_A)$. Then $\varphi^p_A([x]) = [f^p(x)] = [0] \in H^p(B^p)$. Since φ^p_A is a group homomorphism, $\varphi^p_A([0]) = [f^p(0)] = [0]$. This implies $f^p(x) = f^p(0) = 0$, and since f is injective, $x = 0 \in A^p$. Since δ^{p-1} is a homomorphism, we know that for $[0] \in H^{p-1}(C^{p-1})$ we have $\delta^{p-1}([0]) = [0] = [x] \in H^p(A^p)$. Thus $[x] \in im(\delta^{p-1})$ and $ker(\varphi^p_A) = im(\delta^{p-1})$

Let $[b] \in im(\varphi^p_A)$. Then $\varphi^p_A([a]) = [f^p(a)] = [b]$ for some $[a] \in H^p(A^p)$. This implies $f^p(a) = b$, which means $b \in im(f^p)$. Then $b \in ker(g^p) = im(f^p)$. Therefore, $\varphi^p_B([b]) = [g^p(b)] = [0] \in H^p(C^p)$. Thus $[b] \in ker(\varphi^p_B)$ and $im(\varphi^p_A) \subseteq ker(\varphi^p_B)$. Now let $[b] \in ker(\varphi^p_B)$. Well, $\varphi^p_B([b]) = [g^p(b)] = [0]$, which implies $g^p(b) = 0$. Then $b \in ker(g^p) \Rightarrow b \in im(f^p)$. Since $b \in im(f^p)$, there exists some $a \in A^p$ such that $f^p(a) = b$. We already know $[b]$ is a class, so it follows that $[b] = [f^p(a)] = \varphi^p_A([a])$, which means $[b] \in im(\varphi^p_A)$. Therefore $ker(\varphi^p_B) = im(\varphi^p_A)$.

Let $[c] \in im(\varphi^p_B)$. There exists some element $[b] \in H^p(B^p)$ such that $\varphi^p_B[b] = [g^p(b)][c]$. Note that $[b]$ belongs to $H^p(B^p)$ and thus $b \in ker(\partial^p)$. Then we have $g^p(b) = c$ and $\partial^p(b) = 0 \in B^{p+1}$. f^{p+1} is a group homomorphism, so 0 is in its image, and since f^{p+1} is injective, the only element that can map to 0 is the element $0 \in A^{p+1}$. Thus we can map $[c]$ to the class $[0]$ (recall that we showed this is the class $[c]$ maps to regardless of our choice for b). Then $\delta^p([c]) = [0] \in H^{p+1}A^{p+1}$, $[c] \in ker(\delta^p)$, and $im(\varphi^p_B) \subseteq ker(\delta^p)$. Suppose we have some $[x] \in ker(\delta^p)$. Well, since $\delta^p([x]) = [0]$, we have that $0 \in A^{p+1}$. But recall that this maps to some $\delta^p(y) \in B^{p+1}$, where y is an element of B^p such that $g^p(y) = x$. But f^{p+1} is a group homomorphism, so $f^{p+1}(0) = \delta^p(y) = 0 \in B^{p+1}$. Therefore, $y \in ker(\delta^p)$ and $[y] \in H^p(B^p)$. Then $\delta^p_B([y]) = [g^p(y)] = [x]$ and $[x] \in im(\delta^p_B)$. Therefore, $ker(\delta^p) = im(\delta^p_B)$ and the sequence of homology groups is exact.

Homology groups, at first glance, seem tangential to topology. To compute homology groups, the only requirement is that we have a chain complex, which consist not of topological data, but algebraic structures. To define Cech cohomology, we need a way to take a topological space X and generate abelian groups from open sets of X. To understand this idea, we need to understand some basic ideas of category theory.

3. Category Theory

Category theory is the "big picture" of mathematics. In category theory, entire branches of mathematics are thought of as *categories*. In this light, we can attempt to understand "maps" and similarities between different categories. We begin our discussion of category theory by defining categories and looking at some examples of them. Then we will return to the idea of "mapping" between categories.

Definition 3.1. (Categories) A category C is a collection of objects and morphisms having the following properties:

1. Each morphism $f : X \to Y$ has a domain object and codomain object in C
2. Each object X of C has an identity morphism $id_X : X \to X$, which satisfies for any morphism $f : W \to X$ and $g : X \to Y$ the condition that $id_X \circ f = f$ and $g \circ id_X = g$.
3. For any objects $X; Y; Z$ of C and morphisms $f; g$ of C where $f : X \to Y$ and $g : Y \to Z$, we have $g \circ f : X \to Z$. That is, if the codomain of f is equal to the domain of g, then there exists a composite morphism from the domain of f to the codomain of g
4. If $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(Riehl, 3).

Example 3.2. (Category of groups) Consider the collection of objects C of groups and their homomorphisms. Well, we know then that each morphism in C has a domain and codomain of objects in C, otherwise it is not a group homomorphism!

Next we see that each object G of C has an identity morphism. This is just a matter of checking if $id_G : G \to G$ is a homomorphism. Well, if the group G has the binary operation $\ast$, we have that for any $g; h \in G$, $\ast(id_G(g); id_G(h)) = \ast(g; h) = id_G(\ast(g; h))$. Thus, id_G is itself a homomorphism for any group G. Also, for group homomorpisms $f : A \to G$ and $g : G \to B$, we have that $id_G \circ f(a) = id_G(f(a)) = f(a)$ for all $a \in A$ and $g \circ id_G(x) = g(x)$ for all $x \in G$. Therefore $id_G \circ f = f$ and $g \circ id_G = g$.

Now we check that for any objects $X; Y; Z$ of C with homomorphisms $f : X \to Y$ and $g : Y \to Z$ we have the composite morphism $g \circ f : X \to Z$ in C. We will assume that the binary operations on $X; Y;$ and Z are $\ast; \star;$ and $\diamond$ respectively. Then for any $x; x' \in X$ we have $g \circ f(\ast(x; x')) = g \circ \star(f(x); f(x')) = \diamond(g \circ f(x); g \circ f(x'))$, which satisfies the requirements for a group homomorphism.

Lastly, we show that for group homomorphisms $f : A \to B$, $g : B \to C$, and $h : C \to D$ we can compose the functions associatively. Well, $h \circ (g \circ f) = h \circ (g(f(a))) = h \circ (g(f(a))) = h(g(f(a)))$, and $(h \circ g) \circ f(a) = h(g \circ f(a)) = h(g(f(a)))$ for all $a \in A$. It follows that $h \circ (g \circ f) = (h \circ g) \circ f$. Therefore, the collection of groups and their homomorphisms, C, is a category.

Example 3.3. Category of R-modules Consider the collection C of all R-modules and their homomorphisms. With R-module homomorphisms as the designated morphisms for C, we know immediately that each morphism has a domain and codomain in C.

Now we check that the identity map on an R-module A is itself an R-module homomorphism. Let R be the ring of the abelian group A and let $a; b \in A$ and $c \in R$. Then for the identity map $id_A : A \to A$ we have $id_A(a+b) = a+b = id_A(a)+id_A(b)$ and $id_A(c \cdot a) = c \cdot a = c \cdot id_A(a)$. For R-module homomorphisms $f : B \to A$ and $g : A \to C$ we have $(id_A \circ f)(b) = id_A(f(b)) = f(b)$ for all $b \in B$ and $(g \circ id_A)(a) = g(id_A(a)) = g(a)$ for all $a \in A$. Then $id_A \circ f = f$ and $g \circ id_A = g$.

Next we check that for R-modules $X; Y; Z$ with homomorphisms $f : X \to Y$ and $g : Y \to Z$ we have the composite morphism $g \circ f : X \to Z$. Let R be the ring of the abelian group Z and let $x; x' \in X$ and $r \in R$. Then $g \circ f(x + x') = g(f(x) + f(x')) = g \circ f(x) + g \circ f(x')$ and $g \circ f(r \cdot x) = g(r \cdot f(x)) = r \cdot (g \circ f(x))$ as desired.

Finally, we check that the composition of R-module homomorphisms is associative. Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then $h \circ (g \circ f)(a) = h \circ (g(f(a)) = h(g(f(a)))$ and $(h \circ g) \circ f(a) = h(g \circ f(a)) = h(g(f(a)))$ for all $a \in A$. Therefore, we have $h \circ (g \circ f) = (h \circ g) \circ f$ and the collection of R-modules and their homomorphisms is a category.

Definition 3.4. (Functors) Let C and D be categories. A functor is a map $F : C \to D$ such that for each object X of C we have an object F_X in D and for each morphism $f : c \to c'$ of C we have a morphism $F_f : F_c \to F_{c'}$ of D satisfying

(1.) $F_{id_c} = id_{F_c}$

(2.) For a morphism $g : d \to d'$ of C we have $F_{g \circ f} = F_g \circ F_f$.

(Riehl, 13).

Example 3.5. (Forgetful functors) In category theory, a forgetful functor is a functor in which structure of the domain of the functor is "forgotten," or removed in the codomain. One such functor would be a map $F : Mod_R \to Gr$, where Mod_R is the category of R-modules and Gr is the category of groups. We define this functor by taking an R-module and "forgetting" the ring and its operation. This leaves us with a group, which of course would be considered an object in Gr. We now show that this is in fact a functor.

Let A be an object of Mod_R. If we apply the functor F to the object A, we forget the ring B of A as well as the binary operation $B \times A \to A$. Then the object F_A is an abelian group, which by definition is an obejct of Gr.

Let $f : A \to A'$ of be a morphism of Mod_R. After applying F_A and $F_{A'}$, we can create a morphism $F_f : F_A \to F_{A'}$ of Gr. Recall that an R-module homomorphism already satisfies the property for a group homomorphism, but with the additional requirement that $f(r \cdot a) = r \cdot f(a)$ for an element r of the ring and an element a of the abelian group. To create a morphism F_f, we simply take the R-module homomorphism f and forget the latter requirement while maintaining the group homomorphism requirement. From this we see that there is an object F_A of Gr for every object A of Mod_R, as well as a morphism F_f of Gr for every morphism f of Mod_R. Consider F_{id_R} for some object R of Mod_R. It follows that F_{id_R} will be the identity of R, but as a group homomorphism. Then F_{id_R} acts as the identity on the group F_R and $F_{id_R} = id_{F_R}$. For an R-module homomorphisms $f : A \to B$ and $g : B \to C$, we have $F_{g \circ f}$, which acts as the composition $g \circ f$ as a group homomorphism. It follows that $F_{g \circ f} = F_g \circ F_f$, and therefore F is a functor.

A similar forgetful functor would be $Y : Gr \to Set$, where Gr is the category of groups and Set is the category of sets. Y is said to be a forgetful functor since the binary operation on an object of Gr is ignored for the sake of considering the group as a set. Since proving Y is a functor would be nearly identical to the proof for F, we assume for brevity that this is the case.

Example 3.6. (The homology functor) Consider the category Ch, whose objects are chain complexes and morphisms are chain maps. For brevity we assume that Ch is indeed a category, though we could check this easily, similar to the categories we discussed in Examples 3.2 and 3.3. We can create a functor from Ch to the category of groups, Gr, with a functor $F : Ch \to Gr$ such that F_{C_*} is the homology group $H_*(C_*)$. It follows then that each chain complex C_* has an object $H_*(C_*)$ in Gr, since we can define homology groups for any chain complex. For each chain map $f_* : C_* \to D_*$ we have a morphism $F_f : F_{C_*} \to F_{D_*}$ which is given by the induced map on homology groups that we defined in Theorem 2.26. For the identity chain map id_C on a chain complex C, we have F_{id_C}, which is the induced identity map on homology $\phi_* : H_*(C_*) \to H_*(C_*)$. For chain maps $f_* : C_* \to D_*$ and $g_* : D_* \to E_*$, we have the morphism $F_{g \circ f}$, which is a homology map $\phi_* : H_*(C_*) \to H_*(E_*)$ given by $\phi^p([c]) = [(g \circ f)(c)]$. We also have F_f acting as the map $H_*(C_*) \to H_*(D_*)$ and F_g is the map $H_*(D_*) \to H_*(E_*)$. It follows that $(F_g \circ F_f)([c]) = F_g(F_f([c])) = F_g([f(c)]) = [g(f(c))] = [(g \circ f)(c)] = phi_*([c])$. Therefore, F is a functor.

The functors in Examples 3.5 and 3.6 are useful, but recall that our goal was to map from the category of topological spaces (one sees that if we have the morphisms be continuous functions, this is indeed a category) to the category of groups. For Cech cohomology, we can look at functors which satisfy some other useful properties, known as *sheaves*. We turn to sheaves in Section 4, but for now we consider another concept in category theory which will help us define the Cech cohomology.

3.1. **Direct Limit.**

Definition 3.7. (Direct limit) A collection of data A is a *directed system* if it consists of the following data:

1. A *directed set* I which is equipped with a reflexive and transitive binary operation $\leq$ such that each element $i \in I$ has a lower bound.
2. A collection of groups $\{A_i\}_{i \in I}$ which is indexed by the directed set I
3. A collection of group homomorphisms $f_{ij} : A_j \to A_i$ satisfying $f_{ij} \circ f_{jk} = f_{ik}$ for all $i \leq j \leq k$ and $f_{ii} = id_{A_i}$

The direct limit of a directed system, denoted by $\lim(A) = \lim_{i \in I} A_i$ is given by

$$\lim_{i \in I} A_i = \coprod_{i \in I} A_i / \sim$$

where $\sim$ is a relation between A_i and A_j defined by $A_i \sim A_j$ if there exists a lower bound k for $\{i, j\}$ such that $f_{ki}(x_i) = f_{kj}(x_j)$ for some $x_i \in A_i$ and $x_j \in A_j$.

The idea behind the direct limit is that we take the category theory equivalent of the disjoint union of groups in A, modulo a relation that relates the pairs of groups which have lower bounds satisfying a certain condition under the morphisms of A. Essentially, we take "all" of the groups in A and identify all of the elements satisfying the relation $\sim$. Often there are groups at the "tail" which are unchanged by any f_{ij}. These elements are called *stable groups*, and are isomorphic to the limit of the directed system.

To truly define Cech cohomology, we must make use of direct limits, including the previously mentioned property of stable groups. However, the methods required to properly use these tools are unknown to the author, as we shall discuss in further detail in Section 5. For now, we turn our attention to the concept of sheaves: functors which make the Cech cohomology process possible.

4. Sheaves

Cech cohomology is computed on a topological space, but the cohomology groups require a chain complex of algebraic structures such as groups or modules to compute. The Cech Complex uses a sheaf to construct these algebraic structures.

Definition 4.1. (Presheaves and sheaves) Let X be a topological space. A presheaf F on X is a collection of sets satisfying the following properties:

1. For every open set $U \subseteq X$ we have a set $F(U)$. The set $F(\emptyset)$ is assumed to contain only one element.
2. For every inclusion of open sets $V \subseteq U$, we have a map $\rho_{V,U} : F(U) \to F(V)$ called the restriction map, which satisfies for an inclusion $W \subseteq V \subseteq U$ $\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$

We say F is a sheaf if in addition to satisfying the first two properties we have for every open set V of X, every open cover $W = \{W_i\}_{i \in I}$ of V, and every family of sets $\{s_i\}_{i \in I}$ where $s_i \in F(W_i)$ and $\rho_{W_i \cap W_j, W_i}(s_i) = \rho_{W_i \cap W_j, W_j}(s_j)$, there exists a unique $s \in F(V)$ such that $\rho_{W_i, V}(s) = s_i$ (Brylinski, 1-2).

One sees that presheaves are in fact functors. The category of a presheaf's domain is determined by the morphisms of open cover objects $U_i \hookrightarrow U_j$. For our discussions, we assume our presheaves take inputs from the category of topological spaces and land in the category of groups. If F is a presheaf over a topological space X, we have that for an open set $U \subseteq X$, which is itself an object in the category of topological spaces, has an object $F(U)$ in the category of groups. For an inclusion $V \subseteq U$ of open sets, we have that the inclusion map (which is by definition continuous, and therefore a morphism) has a map $F_f = \rho_{V,U} : F(U) \to F(V)$. Thus, by Definition 3.4, a presheaf F is a functor. In the following examples, we look at three different presheaves.

Example 4.2. (The sheaf of continuous functions)

Let X be a topological space and F_X be the collection of sets of continuous functions from X to $\mathbb{R}$. We can show that this is a sheaf. Naturally, then, for an open set $U \subseteq X$ we have a set $F(U) = \{U \to \mathbb{R} \mid f \text{ is continuous}\}$. If we have an inclusion $V \subseteq U$ of open sets of X, we can create map $\rho_{V,U} : F(U) \to F(V)$ by taking a continuous function $f : U \to \mathbb{R}$ and restricting it to V by $f|_V : V \to \mathbb{R}$ which we denote by $\rho_{V,U}(f)$. Note that the restriction map $f|_V$ is itself continuous. For each $v \in V$ and each open neighborhood A of $f|_V(x)$ we have an open set B of U such that $f(B) \subseteq A$, since f is continuous. But since $V \subseteq U$, the set $B \cap V$ is open in V. Thus $f(B \cap V) \subseteq A$ and $f|_V$ is continuous at each point $v \in V$. For a function, $f : X \to Y$ restricted to some set $A \subseteq X$, we have that the restriction of f to A, denoted by $f|_A := (A \times Y) \cap f$, where $f \subseteq X \times Y$. Necessarily, then, for an inclusion of open sets $W \subseteq V \subseteq U$ with a continuous function $f : U \to \mathbb{R}$, we have that $\rho_{W,V} \circ \rho_{V,U}(f) = \rho_{W,V}(f|_V) = (f|_V)|_W = (W \times \mathbb{R}) \cap ((V \times \mathbb{R}) \cap f) = ((W \cap V) \times (\mathbb{R} \cap \mathbb{R})) \cap f = ((W \cap V) \times \mathbb{R}) \cap f = (W \times \mathbb{R}) \cap f = f|_W = \rho_{W,U}$. We see then, that F_X is a presheaf.

Let U be an open set of X, $V = \{V_i\}_{i \in I}$ be an open cover of U, and $\{s_i\}_{i \in I}$ be a family of sets where $s_i \in F(V_i)$ and $\rho_{V_i \cap V_j, V_i}(s_i) = \rho_{V_i \cap V_j, V_j}(s_j)$ for all $i, j \in I$. We wish to show that there is a unique $s \in F(U)$ such that $\rho_{V_i,U}(s) = s_i$ for all $i \in I$. Well, since each s_i is a continuous function $f : V_i \to \mathbb{R}$, and that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for each i, j, we can define a continuous function $s : U \to \mathbb{R}$ as a piecewise function $s(u) := \begin{cases} s_i(u) & u \in V_i \end{cases}$. We know this function will always be defined, since any $u \in U$ must be contained in an open set V_i of the cover V, and the corresponding s_i is continuous, with s_i and s_j agreeing on the intersection $V_i \cap V_j$. It follows then that for any $u \in U$ and any open neighborhood A containing $s(u)$ we have that $u \in V_i$ for some $i \in I$. Then $s(u) = s_i(u)$, which means there exists some open neighborhood $B \subseteq V_i$ such that $s_i(B) \subseteq A$. Since B is open on V_i and $V_i \subseteq U$, it follows that B is also open on U. Then that $s_i(B) = s(B) \subseteq A$, which implies s is continuous at each point $u \in U$. Therefore $s \in F(U)$. It follows by our definition of s that $\rho_{V_i,U}(s) = s|_{V_i} = s_i$ for all $i \in I$.

Lastly, we show that s is unique. Suppose that there was another function $t \in F(U)$ such that $\rho_{V_i,U}(t) = s_i$ for all $i \in I$. Then for any $u \in U$, which must also be contained in some $V_i \in V$, we have that $t(u) = t|_{V_i}(u) = \rho_{V_i,U} \circ t(u) = s_i(u)$. Based on our definition of s, we have that $s(u) = s_i(u) = t(u)$. Therefore, $s = t$, and s is unique. Thus we have that the collection of sets F_X is a sheaf on X. The sheaf is usually denoted by $C^0(X)$, and there are similar sheaves such as $C^1(X) = \{f : X \to \mathbb{R} | f \text{ is differentiable}\}$, $C^2(X) = \{f : X \to \mathbb{R} | f \text{ has continuous second derivatives}\}$, and so on.

Example 4.3. (The presheaf of constant functions is not a sheaf) Let X be a topological space. We can create a presheaf F_X on X by the following: $F_X := \{f : X \to \mathbb{R} | f \text{ is constant}\}$. It follows immediately that for an open set $U \subseteq X$ we have a set $F(U) = \{f : U \to \mathbb{R} | f \text{ is constant}\}$. For an inclusion of open sets $V \subseteq U$, we define our restriction map $\rho_{V,U} : F(U) \to F(V)$ by $\rho_{V,U}(f) = f|_V$, since we know that a constant function on U will have the same value when restricted to the subset V. For an inclusion of open sets $W \subseteq V \subseteq U$ with a constant function $f : U \to \mathbb{R}$, we have $\rho_{W,V} \circ \rho_{V,U}(f) = \rho_{W,V}(f|_V) = (f|_V)|_W = (W \times \mathbb{R}) \cap ((V \times \mathbb{R}) \cap f) = ((W \cap V) \times \mathbb{R}) \cap f = (W \times \mathbb{R}) \cap f = f|_W = \rho_{W,U}$. We see then, that F_X is a presheaf. However, F_X is not a sheaf, as we now show.

Consider the case where X is not a connected space. Then we have a disjoint pair of open sets $U_1, U_2 \subseteq X$ such that $U_1 \cup U_2 = X$. Note that this implies U_1 and U_2 form an open cover of the open set X. Define a constant function $s_1 \in F(U_1)$ by $s_1(u_1) = \sqrt{2}$ and a constant function $s_2 \in F(U_2)$ by $s_2(u_2) = \sqrt{3}$. This yields a family of sets $\{s_1, s_2\}$ where $s_i \in F(U_i)$ for each $i \in \{1, 2\}$. Since $U_1 \cap U_2 = \emptyset$, it follows that $\rho_{U_1 \cap U_2, U_1}(s_1) = \rho_{U_1 \cap U_2, U_2}(s_2)$, since it is vacuously true that $s_1|_\emptyset = s_2|_\emptyset$. Recall that for F_X to be a sheaf, it must guarantee under these conditions the existence of a constant function $s \in F(X)$ such that $\rho_{U_1,X}(s) = s_1$ and $\rho_{U_2,X}(s) = s_2$. But it is impossible for such a function to exist, since s must be constant and s_1 and s_2 are constant functions of different values. Therefore, while F_X is a presheaf, it is not a sheaf.

Example 4.4. (The sheaf of locally constant functions) The most important sheaf in our discussion of Cech cohomology will be the collection of all locally constant functions from a topological space X to the real numbers. This set, which will denote by $\underline{\mathbb{R}}_X$ is defined by $\underline{\mathbb{R}}_X := \{f : X \to \mathbb{R} | f \text{ is locally constant}\}$. For an open set $U \subseteq X$, we have a collection of sets $\underline{\mathbb{R}}_X(U) = \{f : U \to \mathbb{R} | f \text{ is locally constant}\}$, and for an inclusion of open sets $V \subseteq U$ we define the restriction map $\rho_{V,U} : F(U) \to F(V)$ by $\rho_{V,U}(f) = f|_V$. The restriction map ρ will yield yet another locally constant function, since V is just a subset of U, and therefore any element $v \in V$ is an element of U, which means there exists an open neighborhood W of v such that f is constant for all elements of W. Thus $\rho_{V,U}(f)$ will be an element of $R(V)$. For an inclusion of open sets $W \subseteq V \subseteq U$ with a locally constant function $f : U \to \mathbb{R}$, we have $\rho_{W,V} \circ \rho_{V,U}(f) = \rho_{W,V}(f|_V) = (f|_V)|_W = (W \times \mathbb{R}) \cap ((V \times \mathbb{R}) \cap f) = ((W \cap V) \times \mathbb{R}) \cap f = (W \times \mathbb{R}) \cap f = f|_W = \rho_{W,U}$. Then R satisfies the requirements for a presheaf. Now we show that R is a sheaf, unlike the special case of constant functions, which only forms a presheaf.

Suppose that for an open set $U \subseteq X$ we have an open cover $V = \{V_i\}_{i \in I}$ and family of sets $\{s_i\}_{i \in I}$ where $s_i \in R(V_i)$ and $\rho_{V_i \cap V_j, V_i}(s_i) = \rho_{V_i \cap V_j, V_j}(s_j)$ for all $i, j \in I$. We need to show that there exists a unique $s \in R(U)$ such that $\rho_{V_i,U}(s) = s_i$ for all $i \in I$. We can define s as a piecewise function, just as we did in Example 4.2. $s(u) := \begin{cases} s_i(u) & u \in V_i \end{cases}$ will itself be a locally constant function, as we now show. Let $u \in U$. It follows then that $u \in V_i$ for some $i \in I$. Then $s(u) = s_i(u)$. Since s_i is locally constant, there exists some open neighborhood W of u such that $s_i(W)$ is constant. Since W is an open subset of V_i, and V_i is an open subset of U, $s_i(W) = s(W)$ and W is an open subset of U. Thus there exists for each $u \in U$ an open neighborhood W of u such that s is constant, and the function s is locally constant as we desired.

It follows naturally that $\rho_{V_i,U}(s) = s|_{V_i} = s_i$ for all $i \in I$. Such a function exists because our only requirement is that s be *locally* constant. This means that even if X is disjoint, as in the counterexample for Example 4.3, s can take different values and still belong to the set $R(U)$. Lastly, suppose that there exists a locally constant function $t \in R(U)$ satisfying $\rho_{V_i,U}(t) = s_i$ for all $i \in I$. Well, for some $u \in U$, which must also be contained in an open set $V_i \in V$, we have that $t(u) = t|_{V_i}(u) = \rho_{V_i,U} \, t(u) = s_i(u)$. But based on our definition of s we also have $s(u) = s_i(u) = t(u)$. Thus, s is unique, and we have shown that the collection R of locally constant functions from the topological space X to the real numbers is a sheaf.

Recall from Example 2.13 that locally constant functions from a topological space to the real numbers form a vector space. It follows then that the sheaf $\underline{\mathbb{R}}_X$ over a topological space X is a functor which takes topological spaces and maps them to R-modules. We have also discussed that locally constant functions determine the structure of a topological space, and as such they will be the most beneficial sheaf for us to use. Now that we understand sheaves and presheaves, we may now formally define Cech cohomology.

5. Cech cohomology

The main idea of Cech cohomology is forming a chain complex from a topological space. If we take an open cover of a topological space, and we apply a sheaf of abelian groups to each set of the cover, we can generate a special kind of chain complex. We begin defining Cech cohomology by defining this chain complex and its homomorphisms.

Definition 5.1. (The Cech complex) Let A be a presheaf of abelian groups over a topological space X with open cover $U = \{U_i\}_{i \in I}$. A Cech complex is generated by the sections of A applied to finite intersections of elements of U. Let $U_{i_0,\dots,i_p}$ denote the p-fold intersection of open sets of U. That is, for $(i_0, \dots, i_p) \in I$, $U_{i_0,\dots,i_p} = U_{i_0} \cap \dots \cap U_{i_p}$, where each $U_i \in U$. The p-th set in the Cech complex is denoted by

$$C^p(U; A) = \prod_{i_0,\dots,i_p} A(U_{i_0,\dots,i_p})$$

Elements of $C^p(U; A)$ are families or "sequences" of elements $\alpha = \alpha_{i_0,\dots,i_p} \in A(U_{i_0,\dots,i_p})$. The homomorphisms in the Cech complex are given by

$$\delta^p(\alpha)_{i_0,\dots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho(\alpha_{i_0,\dots,i_{j-1},i_{j+1},\dots,i_{p+1}})|_{U_{i_0,\dots,i_{p+1}}}$$

(Brylinski, 25).

Example 5.2. (For the homomorphism of the Cech complex, $\delta^{p+1}(\delta^p) = 0$)

For the p-th set in a Cech complex, we have that the homomorphism $\delta^p : C^p \to C^{p+1}$ is given by

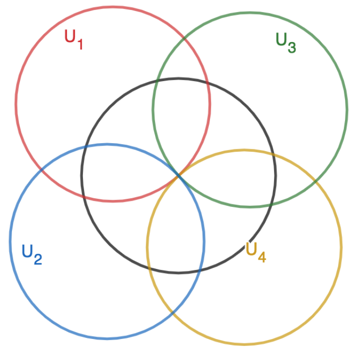
$$\delta^p(\alpha)_{i_0,\dots,i_p} = \sum_{j=0}^{p+1} (-1)^j \rho(\alpha_{i_0,\dots,\hat{i_j},\dots,i_{p+1}})|_{U_{i_0,\dots,i_{p+1}}},$$

where α is a family of sets in the sheaf A applied to the p-fold intersection $U_{i_0,\dots,i_p}$. Then

$$\delta^{p+1} \circ \delta^p(\alpha) = \delta^{p+1}\Big(\sum_{j=0}^{p+1} (-1)^j \rho(\alpha_{i_0,\dots,\hat{i_j},\dots,i_{p+1}})\Big)$$

$$= \sum_{k=0}^{p+2} (-1)^k \rho\Big(\sum_{j=0}^{p+1} (-1)^j \rho(\alpha_{i_0,\dots,\hat{i_j},\dots,i_{p+1}})\Big)_{i_0,\dots,\hat{i_k},\dots,i_{p+2}}$$

$$= \sum_{k=0}^{p+2} \sum_{j=0}^{p+1} (-1)^{j+k} \rho(\rho_{i_0,\dots,\hat{i_j}\dots,i_{p+1}})_{i_0,\dots,\hat{i_k},\dots,i_{p+2}}$$



(A) Our topological space, S^1

(B) S^1 with ϵ-balls

FIGURE 4. S^1 on its own (A) and with ϵ-balls (B).

Note that this means we cannot have $j = k$, as in this case we cannot remove the k-th index if we have already removed the j-th index. Then either $j < k$ or $k < j$. Adjusting our summations to account for $j \neq k$ yields

$$\delta^{p+1}(\sigma^p) = \sum_{j<k}^{p+2} (-1)^{j+k}(\sigma_{i_0,\dots,\hat{i_j},\dots,\hat{i_k},\dots,i_{p+2}}) + \sum_{k<j}^{p+2} (-1)^{j+k}(\sigma_{i_0,\dots,\hat{i_k},\dots,\hat{i_j},\dots,i_{p+2}})$$

But note that regardless of whether $j < k$ or $k < j$, we still remove the j-th index first. However, the case where $j < k$ affects the indexing set differently from the case when $k < j$. When $k < j$, removing j first has no influence on our removal of the k-th index. But when $j < k$, every index greater than j is "shifted to the left," since we remove the j-th index first, and thus the k-th index then becomes what was originally the $k+1$-th item in the indexing set.

For example, we see that $k = 0$, $j = 2$ corresponds to

$$(-1)^{2+0}(\sigma_{i_1,i_3,\dots,i_{p+2}}) = \sigma_{i_1,i_3,\dots,i_{p+2}}$$

but $j = 0$, $k = 1$ corresponds to

$$(-1)^{0+1}(\sigma_{i_1,i_3,\dots,i_{p+2}}) = -\sigma_{i_1,i_3,\dots,i_{p+2}}$$

These terms will of course cancel out when added together. However, before we can generalize these results, we must adjust our summations yet again, as the current notation does not allow us to remove $j = i_0$ and $k = i_1$. To adjust this, we simply allow for $j \leq k$. It follows that, under this notation, each term will appear in each sum, but with alternate signs. Thus

$$\delta^{p+1}(\sigma^p) = \sum_{j \leq k}^{p+2} (-1)^{j+k}(\sigma_{i_o,\dots,\hat{i_j},\dots,\hat{i_k},\dots,i_{p+2}}) + \sum_{k<j}^{p+2} (-1)^{j+k}(\sigma_{i_0,\dots,\hat{i_k},\dots,\hat{i_j},\dots,i_{p+2}}) = 0$$

Once we have a Cech complex, we may compute the cohomology groups for our topological space. In practice, computing the Cech cohomology can be done through linear algebra. For the next few examples, we compute the Cech cohomology for topological spaces and check the results against our intuition.

Example 5.3 (Cech cohomology of the circle, S^1)**.** Suppose we have a circle in $\mathbb{R}^2$. When we compute the Cech cohomology of a shape, we have a distinct advantage as opposed to when we work with some random data set. The advantage is that when taking the Cech cohomology of a shape, we can intuitively predict what our cohomology groups will be. Looking at Figure 1A, we can see that there is one cluster of points: the points on the edge of the circle. We also see that there is a loop here as well - the circle itself. These observations correspond to $H^0 = 1$ and $H^1 = 1$ respectively, and we can check our computations against them.

We can cover S^1 with four ϵ-balls in $\mathbb{R}^2$, as seen in Figure 1B. Because the ϵ-balls in this example are also circles, they have been colored to prevent confusion. We could use the ϵ-balls to determine intersections, but when applicable, it is always better to use the actual cover of the space to determine intersections. For S^1,

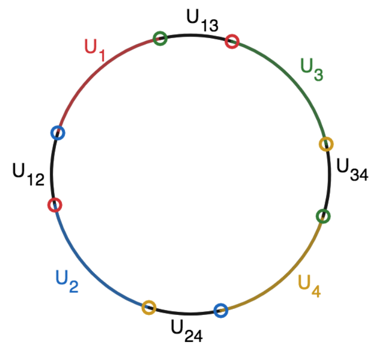


FIGURE 5. The cover for S^1 along with all of the intersections.

we can take our four ϵ-balls and intersect them with S^1 and turn our ϵ-spheres into four arcs. These arcs, shown in Figure 2, make it easier to determine intersections. Any overlapping arcs are said to intersect one another.

Although we can ignore most of the theory behind Cech cohomology when applying it practically, to familiarize the theory, this first exercise in Cech cohomology will outline how we apply the formal definitions to our problem. Exercises 5.4 and 5.5, however, will focus on computations instead.

Moving on to the Cech Complex, we see from Figure 2 that we have used four open sets to cover S^1, and each pair of sets intersects for a total of four intersections. Since we have no higher intersections, we know that for $p > 1$, $C^p = 0$ in our Cech Complex. Since we have four open sets, we know that

$$C^0 = \prod_{i \in \{1,2,3,4\}} R(U_i) = \bigoplus_{i \in \{1,2,3,4\}} R(U_i) = R(U_1) \oplus R(U_2) \oplus R(U_3) \oplus R(U_4)$$

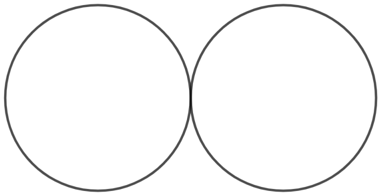
where $R(U_i)$ is the module of locally constant functions from U_i to $\mathbb{R}$. Similarly,

$$C^1 = \prod_{i,j \in \{1,2,3,4\}} R(U_{ij})$$

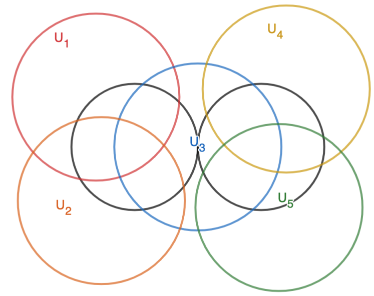
where U_{ij} represents the intersection $U_i \cap U_j$ and $i < j$. Thus, our Cech Complex is defined by $C^\bullet := 0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0$. Note that δ^{-1} and δ^1 are not necessary to define in this scenario (the former is in fact never necessary to define). As for the basis elements of our sets, elements of C^0 are in the form (f_1, f_2, f_3, f_4) and elements of C^1 are represented by $(f_{12}, f_{13}, f_{23}, f_{24})$. Note that while the product C^1 is not ordered, we do not need to consider f_{11}, f_{22}, and so on. We can ignore the functions f_{ii} since they belong to the intersection $U_{ii} = U_i$ and thus have already been accounted for in C^0. To compute our cohomology groups, we only need $|im(\delta^{-1})|$ and $|ker(\delta^{1})|$, both of which we can see are 0.

Before we can compute the cohomology groups for S^1, we need to define $\delta^0 : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\delta^0 : \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \mapsto \begin{pmatrix} f_{12} \\ f_{13} \\ f_{23} \\ f_{34} \end{pmatrix}$. Recall that this is accomplished by defining on each nonempty double intersection U_{ij}

$$\delta^0(f_1, f_2, f_3, f_4)_{U_{ij}} = f_j - f_i$$



(A) The topological space $S^1 \wedge S^1$



(B) $S^1 \wedge S^1$ covered by ϵ-balls

FIGURE 6. $S^1 \wedge S^1$ on its own (A) and with ϵ-balls (B).

In practice, this corresponds to the matrix $\delta^0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. To find $|im(\delta^0)|$, we reduce the matrix to $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, which tells us that $|im(\delta^0)| = rank(\delta^0) = 3$. It follows, then, that $|ker(\delta^0)| = |\mathbb{R}^4| - |im(\delta^0)| = 4 - 3 = 1$. Now we may begin computing the cohomology groups for S^1.

Starting with H^0, we have that $H^0(S^1; U) = ker(\delta^0)/im(\delta^{-1}) = \mathbb{R}^{|ker(\delta^0)| - |im(\delta^{-1})|} = \mathbb{R}^{1-0} = \mathbb{R}^1$. Thus, we say $H^0 = 1$, just as we predicted. Similarly, $H^1(S^1; U) = ker(\delta^1)/im(\delta^0) = \mathbb{R}^{|ker(\delta^1)| - |im(\delta^0)|} = \mathbb{R}^{4-3} = \mathbb{R}^1$. We have that $H^1 = 1$, which again lines up with our intuition.

Note that in the previous example, we did not use a definite ϵ for the radius of our ϵ-balls. For a shape, we need not worry about the size of ϵ, but rather how many ϵ-balls we use. If, for instance, we tried to cover S^1 with just one ϵ-ball, our cover for S^1 would be a circle. As discussed in Example 1.20, circles are not contractible. Our Cech Complex will likely be different if we use a different ϵ, though, as we may need more ϵ-balls if our ϵ is smaller; we may have higher intersections if our ϵ is larger. Also note that we showed the Cech cohomology for *any* circle. When we are working with spaces, the cohomology groups will be the same regardless of its size. A circle with a radius of 100 meters will have the same Cech cohomology of a circle with a radius of 1 meter.

Example 5.4. (Cech cohomology of wedged circles) Consider the set $S^1 \wedge S^1$, which consists of two circles tangent to each other as pictured in Figure 3A. Computing the Cech cohomology of this set requires us to first generate an open cover, which we obtain by taking an open cover of a superset ($\mathbb{R}^2$) and intersecting it with $S^1 \wedge S^1$, as depicted in Figures 3B and 4 respectively. Notice that there is an open set U_3 centered at the point of tangency for $S^1 \wedge S^1$ which covers a significant portion of both circles. The shape formed by U_3, as seen in Figure 4, is still contractible, and ensures that elements of $S^1 \wedge S^1$ are covered with the optimal number of open sets to reduce calculation. Intuitively, we can expect $H^0 = 1$, since the set $S^1 \wedge S^1$ is connected via the point of tangency. We can also predict that $H^1 = 1$, because even though we have two loops, they are connected at a single point. For any $p > 1$, we can expect $H^p = 0$, since there are evidently no higher dimensional loops or components.

Now to define our Cech Complex. Figure 4 shows us that we have used 5 open sets to cover $S^1 \wedge S^1$. Our arrangement of the 5 open sets yielded 6 double intersections, with no higher intersections. Recall from Example 5.3 that this means for $p > 1$, we have $C^p = \{0\}$. Defining the nontrivial R-modules yields

$$C^0 = \prod_{i \in \{1,2,3,4,5\}} R(U_i) = \bigoplus_{i \in \{1,2,3,4,5\}} R(U_i)$$

$$C^1 = \prod_{i,j \in \{1,2,3,4,5,6\}} R(U_{ij}) = \bigoplus_{i,j \in \{1,2,3,4,5,6\}} R(U_{ij})$$

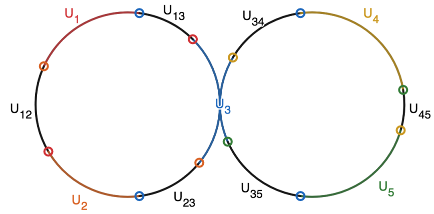


FIGURE 7. The open cover for $S^1 \wedge S^1$

Therefore our Cech Complex is given by $C^\bullet(S^1 \wedge S^1; \mathcal{U}) = (0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0)$, where $\mathcal{U}$ is the open cover $\{U_i\}_{i \in 1,2,3,4,5}$. Just as with Example 5.3, there is no need to define δ^{-1} and δ^1, since we already know that $|im(\delta^{-1})| = 0$ and can see easily that $|ker(\delta^1)| = |C^1| = 6$, which is the only information we need to know regarding those maps. We still need a map

$$\delta^0 : \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} \mapsto \begin{pmatrix} f_{12} \\ f_{13} \\ f_{23} \\ f_{34} \\ f_{35} \\ f_{45} \end{pmatrix},$$

where $(f_1, f_2, f_3, f_4, f_5)$ is a basis for C^0 and $(f_{12}, f_{13}, f_{23}, f_{34}, f_{35}, f_{45})$ is a basis for C^1. Based on our definition of δ^p, the desired map is

$$\delta^0(f_1, f_2, f_3, f_4, f_5)_{ij} = \sum_{k=0}^{1} (-1) f_{i,\hat{k},j}|U_{ij} = f_j - f_i$$

where ij is the ij-th component of an element in C^1. This corresponds to the matrix $\delta^0 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$

After Gauss-Jordan reduction, δ^0 becomes $\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, which implies $|im(\delta^0)| = rank(\delta^0) = 4$.

Thus $|ker(\delta^0)| = |C^0| - |im(\delta^0)| = 5 - 4 = 1$.

Now we may begin computing the cohomology groups. Just as we predicted,

$$H^0(\mathcal{U}; \mathbb{R}) = ker(\delta^0)/im(\delta^{-1}) = \mathbb{R}^{|ker(\delta^0)| - |im(\delta^{-1})|} = \mathbb{R}^{1-0} = \mathbb{R}^1$$

Also in line with our intuition, we have

$$H^1(\mathcal{U}; \mathbb{R}) = ker(\delta^1)/im(\delta^0) = \mathbb{R}^{|ker(\delta^1)| - |im(\delta^0)|} = \mathbb{R}^{6-4} = \mathbb{R}^2$$

Example 5.5. (Cech cohomology of the torus) Consider the torus, $T^2 := S^1 \times S^1$ as a subset of $\mathbb{R}^3$. We see that the torus is one connected piece of data, and that it has two loops: the loop going "sideways" across the torus, and the loop running "vertically" around the ring of the torus. Furthermore, there is the obvious 3-dimensional loop in the center of the torus. From these observations we can expect $H^0 = 1$, $H^1 = 2$, and $H^2 = 1$, with higher degree homology groups yielding 0. Since open sets of the torus can be difficult to visualize we can use the fact that T^2 and its topology come from the ***quotient topology*** of the map $q : I^2 \to T^2$, where $I = [0, 1]$. We do this by identifying all the elements of I^2 satisfying $(x, 0) \sim (x, 1)$

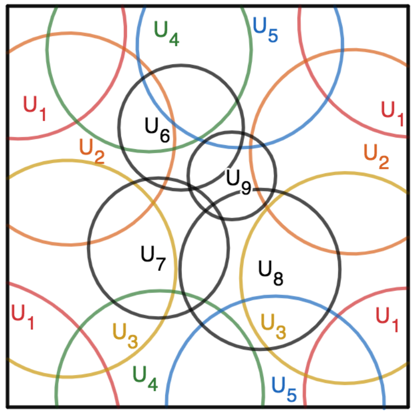


FIGURE 8. The open cover of I^2, which is also an open cover of T^2

and $(0,y) \sim (1,y)$. It follows that any open cover of I^2 will have an open cover on T^2. Figure 9 shows an open cover of I^2.

From the figure, we see that we have 9 open sets, which means

$$C^0 = \prod_{i \in 1,\dots,9} R(U_i)$$

has a basis of 9 elements. Our open sets yield 27 intersections, and so

$$C^1 = \prod_{i,j \in 1,\dots,27} R(U_{ij})$$

This time, we have several three-fold intersections. From Figure 8, we see there are 18 triple intersections, and so

$$C^2 = \prod_{i,j,k \in 1,\dots,18} R(U_{ijk})$$

Thus our Cech complex is

$$0 \xrightarrow{\delta_{-1}} C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} 0$$

It follows that our matrix ^{0} will map

$$\begin{array}{c} 0 \quad 1 \\ \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{pmatrix} \end{array} \mapsto \begin{array}{c} 0 \quad 1 \\ \begin{pmatrix} f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{23} \\ f_{24} \\ f_{25} \\ f_{26} \\ f_{27} \\ f_{28} \\ f_{29} \\ f_{34} \\ f_{35} \\ f_{37} \\ f_{38} \\ f_{45} \\ f_{46} \\ f_{47} \\ f_{48} \\ f_{56} \\ f_{58} \\ f_{59} \\ f_{67} \\ f_{69} \\ f_{78} \\ f_{79} \\ f_{89} \end{pmatrix} \end{array}$$

which corresponds to the matrix

$$
{}^{0} := \begin{array}{c|ccccccccc}
 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 \\
\hline
f_{12} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{13} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{14} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
f_{15} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
f_{23} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
f_{24} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
f_{25} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
f_{26} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
f_{27} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
f_{28} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
f_{29} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
f_{34} & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
f_{35} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
f_{37} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
f_{38} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
f_{45} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
f_{46} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
f_{47} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
f_{48} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
f_{56} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
f_{58} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
f_{59} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
f_{67} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
f_{69} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
f_{78} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
f_{79} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
f_{89} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

The matrix ^{0} reduces to

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	
1	0	0	0	0	0	0	0	1	f_{12}
0	1	0	0	0	0	0	0	1	f_{13}
0	0	1	0	0	0	0	0	1	f_{14}
0	0	0	1	0	0	0	0	1	f_{15}
0	0	0	0	1	0	0	0	1	f_{23}
0	0	0	0	0	1	0	0	1	f_{24}
0	0	0	0	0	0	1	0	1	f_{25}
0	0	0	0	0	0	0	1	1	f_{26}
0	0	0	0	0	0	0	0	0	f_{27}
0	0	0	0	0	0	0	0	0	f_{28}
0	0	0	0	0	0	0	0	0	f_{29}
0	0	0	0	0	0	0	0	0	f_{34}
0	0	0	0	0	0	0	0	0	f_{35}
0	0	0	0	0	0	0	0	0	f_{37}
0	0	0	0	0	0	0	0	0	f_{38}
0	0	0	0	0	0	0	0	0	f_{45}
0	0	0	0	0	0	0	0	0	f_{46}
0	0	0	0	0	0	0	0	0	f_{47}
0	0	0	0	0	0	0	0	0	f_{48}
0	0	0	0	0	0	0	0	0	f_{56}
0	0	0	0	0	0	0	0	0	f_{58}
0	0	0	0	0	0	0	0	0	f_{59}
0	0	0	0	0	0	0	0	0	f_{67}
0	0	0	0	0	0	0	0	0	f_{69}
0	0	0	0	0	0	0	0	0	f_{78}
0	0	0	0	0	0	0	0	0	f_{79}
0	0	0	0	0	0	0	0	0	f_{89}

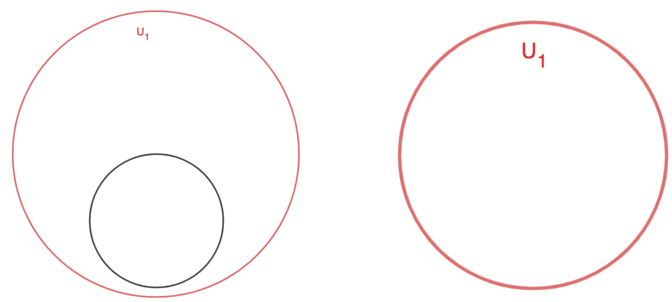
which means $rank(\ ^0) = |im(\ ^0)| = 8$ and implies $|ker(\ ^0)| = |C^0| \quad |im(\ ^0)| = 9 \quad 8 = 1$. For the matrix ^{1}, we wish to map

$$\begin{pmatrix} f_{12} \\ f_{13} \\ f_{14} \\ f_{15} \\ f_{23} \\ f_{24} \\ f_{25} \\ f_{26} \\ f_{27} \\ f_{28} \\ f_{29} \\ f_{34} \\ f_{35} \\ f_{37} \\ f_{38} \\ f_{45} \\ f_{46} \\ f_{47} \\ f_{48} \\ f_{56} \\ f_{58} \\ f_{59} \\ f_{67} \\ f_{69} \\ f_{78} \\ f_{79} \\ f_{89} \end{pmatrix} \mapsto \begin{pmatrix} f_{124} \\ f_{125} \\ f_{134} \\ f_{135} \\ f_{237} \\ f_{238} \\ f_{246} \\ f_{259} \\ f_{267} \\ f_{289} \\ f_{347} \\ f_{358} \\ f_{456} \\ f_{458} \\ f_{478} \\ f_{569} \\ f_{679} \\ f_{789} \end{pmatrix}$$

which yields the matrix

$^1 :=$

	f_{12}	f_{13}	f_{14}	f_{15}	f_{23}	f_{24}	f_{25}	f_{26}	f_{27}	f_{28}	f_{29}	f_{34}	f_{35}	f_{37}	f_{38}	f_{45}	f_{46}	f_{47}	f_{48}	f_{56}	f_{58}	f_{59}	f_{67}	f_{69}	f_{78}	f_{79}	f_{89}
f_{124}	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
f_{125}	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
f_{134}	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
f_{135}	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
f_{237}	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
f_{238}	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
f_{246}	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
f_{259}	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
f_{267}	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
f_{289}	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
f_{347}	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
f_{358}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0
f_{456}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0	0	0
f_{458}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0
f_{478}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0
f_{569}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0
f_{679}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0
f_{789}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1



(A) The topological space S^1 (B) S^1 covered by a single -ball (C) The resulting open cover

FIGURE 9. S^1 on its own (A), covered with an -ball (B), and with an open cover (C).

which we can reduce to

	f_{12}	f_{13}	f_{14}	f_{15}	f_{23}	f_{24}	f_{25}	f_{26}	f_{27}	f_{28}	f_{29}	f_{34}	f_{35}	f_{37}	f_{38}	f_{45}	f_{46}	f_{47}	f_{48}	f_{56}	f_{58}	f_{59}	f_{67}	f_{69}	f_{78}	f_{79}	f_{89}
f_{124}	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
f_{125}	0	1	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0
f_{134}	0	0	1	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	1	0	0	0	0	1	1
f_{135}	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
f_{237}	0	0	0	0	0	1	0	0	0	0	1	0	0	1	1	0	0	0	1	0	1	1	0	0	0	1	1
f_{238}	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
f_{246}	0	0	0	0	0	0	0	1	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	1
f_{259}	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1
f_{267}	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
f_{289}	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	1	1
f_{347}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0
f_{358}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0
f_{456}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	0	1	0	0	0
f_{458}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1
f_{478}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0
f_{569}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0
f_{679}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
f_{789}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

It follows that $rank(\ ^1) = jim(\ ^1)j = 17$ and $jker(\ ^1)j = jC^1j \quad jim(\ ^1)j = 27 \quad 17 = 10$. Fortunately, we do not need to define any other matrices, since we see that from the Cech Complex that $jker(\ ^2)j = jC^2j = 18$. Thus our cohomology groups are

$$H^0(U; \mathsf{R}) = ker(\ ^0)/im(\ \ ^1) = \mathsf{R}^{jker(\ ^0)j \ \ jim(\ \ ^1)j} = \mathsf{R}^{1 \quad 0} = \mathsf{R}$$

$$H^1(U; \mathsf{R}) = ker(\ ^1)/im(\ ^0) = \mathsf{R}^{jker(\ ^1)j \ \ jim(\ ^0)j} = \mathsf{R}^{10 \quad 8} = \mathsf{R}^2$$

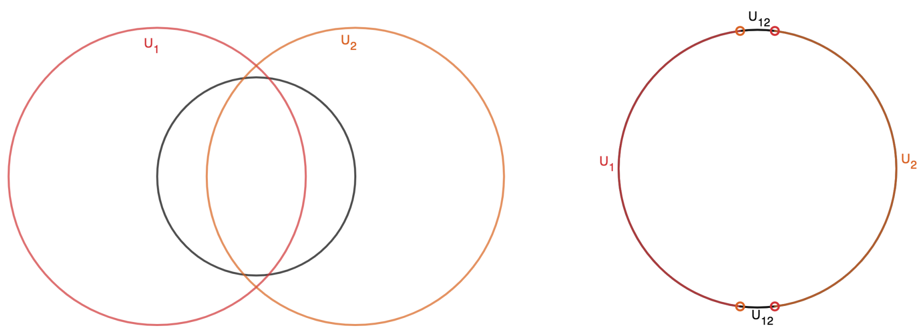
$$H^2(U; \mathsf{R}) = ker(\ ^2)/im(\ ^1) = \mathsf{R}^{jker(\ ^2)j \ \ jim(\ ^1)j} = \mathsf{R}^{18 \quad 17} = \mathsf{R}:$$

Recall that since cohomology groups tell us about what types of holes are present in a space, $H^2 = \mathsf{R}$ corresponds to one 3-dimensional hole in the space.

So far, our Cech Complexes have only had two requirements: that X be a topological space, and that the cover U of X is an open cover. However, not every open cover yields accurate results from the cohomology groups. We can see from the previous examples that if we were to change the number of sets in our open cover, each set in the Cech Complex will have different basis elements, and therefore the cohomology groups will be different. As it turns out, this gives need for another requirement of the open cover U, as we shall now demonstrate.

Example 5.6. (Cech cohomology of a circle, but with bad covers)

Consider the circle S^1. We know from Example 5.3 that we should arrive at $H^0 = 1$ and $H^1 = 1$. However, suppose we wanted to minimize calculation. Our requirement is that the cover of S^1 must be comprised of open sets, so if we choose a large enough open set of R^2, we can intersect it with S^1 and get all of S^1, as shown in Figure 9C. Since S^1 is obviously covered by S^1, and S^1 is a topological space, we should now be



(A) S^1 covered by two ε-balls

(B) The resulting open cover

FIGURE 10. S^1 on its own (A) and with ε-balls (B).

able to construct our Cech Complex using just one open set. This implies $C^p = \{0\}$ for all $p > 0$, and that $C^0 = R(U_1)$. The Cech Complex is then $0 \xrightarrow{\delta^1} C^0 \xrightarrow{\delta^0} 0$, which is certainly easier to work with than the complexes in the previous examples. But notice this already implies $H^1 = 0$, which deviates from both our intuition and our calculations in Example 5.3. A map δ^1 would have as both its domain and codomain 0, meaning $|ker(\delta^1)| = 0$. Similarly, we see from the Cech Complex that $|ker(\delta^0)| = |C^0| = 1$, which implies $|im(\delta^0)| = |C^0| - |ker(\delta^0)| = 1 - 1 = 0$. Therefore, $H^1(U; R) = ker(\delta^1)/im(\delta^0) = R^{|ker(\delta^1)| - |im(\delta^0)|} = R^{0-0} = R^0 = \{0\}$.

The problem is not that we have used one open set to cover S^1, because even with multiple open sets, a cover can still lead to incorrect calculations.

Suppose that we cover S^1 with two open sets of R^2, as shown in Figure 10A. These two open sets will certainly cover S^1, and even result in an intersection, which means C^1 will be nonempty. We have that $C^0 = R(U_1) \oplus R(U_2)$ and $C^1 = R(U_{12})$, which results in the Cech Complex $C^*(S^1; U) := 0 \xrightarrow{\delta^1} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0$. We see that $|ker(\delta^1)| = |C^1| = 1$. δ^0 only needs to map $(f_1, f_2) \mapsto f_{12}$, so it follows naturally that our matrix is $\delta^0 = \begin{pmatrix} -1 & 1 \end{pmatrix}$. There is no possibility of reducing δ^0, so we have $|im(\delta^0)| = rank(\delta^0) = 1$, which implies $|ker(\delta^0)| = |C^0| - |im(\delta^0)| = 2 - 1 = 1$. Then our cohomology groups are

$$H^0(U; R) = ker(\delta^0)/im(\delta^{-1}) = R^{|ker(\delta^0)| - |im(\delta^{-1})|} = R^{1-0} = R^1$$

which aligns with both our intuition and our prior calculations. However, we also have

$$H^1(U; R) = ker(\delta^1)/im(\delta^0) = R^{|ker(\delta^1)| - |im(\delta^0)|} = R^{1-1} = R^0 = \{0\}$$

which contradicts the results from Example 5.3.

The reason for failure is not that we have simply used too few open sets in our cover, but rather that the open sets we used failed a certain contractibility requirement. When we used only one open set in our cover, the open set we obtained, S^1, was not contractible. As we discussed in Example 1.21, S^n is not contractible for any n. In the other scenario, we used two open covers on S^1. Each cover was contractible, but looking at Figure 11, we see that the intersection U_{12} is in two different pieces, which implies that it is impossible for U_{12} to be contractible. Not only must the open sets in our cover be contractible, their intersections must also be contractible.

Theorem 5.7. *If X is a contractible space, then for any open cover U we have $H^p(U; R) = 0$ for all $p > 0$*

The formal proof for Theorem 5.7 is currently beyond the author's understanding. However, intuitively, we can see that if a space X is contractible, it must be connected. Otherwise, the separations of X cannot be continuously deformed to a single point in X without leaving the set. Since X is connected, we can expect X to contain only one "cluster" of points, which corresponds to $H^0(U; R) = 1$. This implies that there is at least one nonzero homology group of X, and that $p = 0$ is not to be the base case of our proof. Since X is contractible, for any point $c \in X$, we can continuously deform X to the point c without leaving X. This implies there are no loops or holes in X, because if there were, we could not deform X to any point. The lack of loops implies that $H^p = 0$ for $p > 0$.

FIGURE 11. The pairwise intersection of covers from Figure 10B

Theorem 2.7 reveals that whatever requirement is needed does not have to come from our topological space. All of the examples we used to compute Cech cohomology were not contractible, yet we still obtained reasonable results for our homology groups. It follows that we need to have a stronger requirement for our open covers, which we define as follows:

Definition 5.8. (Good cover) A good cover U of a topological space X is an open cover for which any finite intersection of any number of sets in U is contractible. That is, an open cover $U = fU_i g_{i2I}$ of a topological space X is a good cover if any p-fold intersection of open sets $U_{i_0;:::;i_p}$ of U is contractible.

One can check that in Example 5.3 we used an open cover that was contractible, while in Example 5.6, neither of the two open covers were contractible. To minimize computations, then, we need to find a good cover with the least number of open sets. But before we can accomplish this, we need to show that good covers of the same topological space will always yield the same cohomology groups.

Theorem 5.9. *If U is a good cover of X and V is a re nement of U then $H^p(U;R) = H^p(V;R)$*

The proof for Theorem 2.9 is also beyond the author's current understanding.

Theorem 2.9 tells us that any pair of good covers for the same topological space will have cohomology groups of the same dimension. We know that if we use too few covers, our cover may not be contractible. So if we wish to minimize calculation, we need to find the minimum number of open sets to create a good cover. To accomplish this, we will need to utilize the direct limit.

5.1. Understanding Cech Cohomology as a Direct Limit.

Theorem 5.10. *If U is any cover of X such that whenever V is a re nement of U we have $H^p(U;R) = H^p(V;R)$ then*

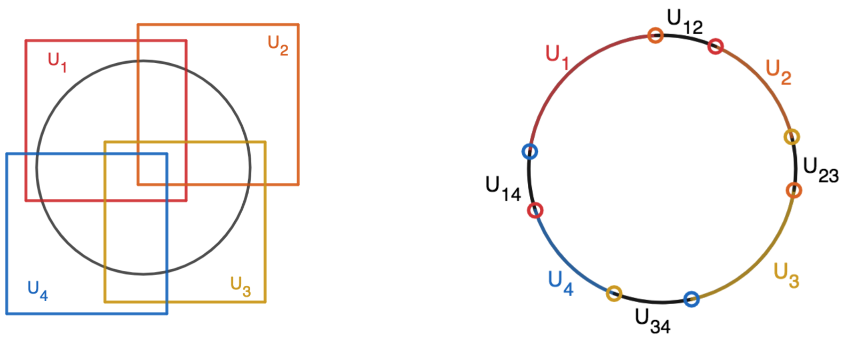
$$\lim_{C}(C;R) = H^p(U;R)$$

Theorem 2.10 says that for a good cover U, which by Theorem 2.9 satisfies the requirement that whenever V is a refinement of U we have $H^p(U;R) = H^p(V;R)$, then the direct limit of all of the covers of the topological space is isomorphic to the stable group $H^p(U;R)$. The cover U, since it satisfies $H^p(U;R) = H^p(V;R)$ for any refinement V, is then the good cover with the fewest number of open sets. While the theorem did not specify that U had to be *good*, we would need to show that only good covers satisfy the conditions required for the refinement. The proof for Theorem 2.10 cannot be proven with the rudimentary concepts of topology, algebra, and category theory outlined previously. The advanced methods required for the proof are beyond the scope of the author, and consequently, cannot be included in the paper.

In closing, we will discuss some interesting concepts that relate to creating an algorithm for the Cech cohomology. Much of the homological algebra from Section 2.1 will be required. Additionally, we will not be able to use -balls to generate open covers, as intersections of two or more -balls is difficult to determine.

5.2. Cech Cohomology in Practice.

So far, we have only used -balls to create open covers. However, so long as we have a good cover, we may use any shape we wish. Recall that n-cubes are also open on R^n. It is easy to check that n-cubes are contractible, since just like -balls, they are "filled in" and can thus be contracted to a point without leaving the set. As it turns out, if we wish to create an algorithm that



(A) S^1 covered by n-cubes

(B) The resulting open cover

FIGURE 12. S^1 covered by n-cubes of $\mathbb{R}^2$ (A) and the open cover formed by the intersection of the n-cubes with S^1 (B).

computes the Cech cohomology of a data set, we will need to make extensive use of n-cubes. First we outline that n-cubes do in fact yield the same results as ϵ-balls.

Example 5.11. (Cech cohomology of a circle, but with cubes)

Consider the topological space S^1, whose cohomology groups we found in Exercise 5.3 to be $H^0 = 1$ and $H^1 = 1$. If we cover S^1 using n-cubes as shown in Figure 12A, we can obtain the open cover shown in Figure 12B, which is unsurprisingly similar to the open cover from Exercise 5.3. It follows from the figure that

$$C^0 = \prod_{i \in 1,2,3,4} R(U_i) = R(U_1) \oplus R(U_2) \oplus R(U_3) \oplus R(U_4)$$

$$C^1 = \prod_{i,j \in 1,2,3,4} = R(U_{12}) \oplus R(U_{14}) \oplus R(U_{23}) \oplus R(U_{34})$$

$$C^*(S^1;U) = 0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0$$

The map δ^0 is given by

$$\delta^0(f_1,f_2,f_3,f_4)_{ij} = \sum_{k=0}^{1} (-1)^k (f_1,f_2,f_3,f_4)_{i,\hat{k},j} = f_j - f_i$$

This corresponds to the matrix $\delta^0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$, which we can reduce to $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Thus, $|im(\delta^0)| = rank(\delta^0) = 3$, which implies $|ker(\delta^0)| = |C^0| - |im(\delta^0)| = 4 - 3 = 1$. Based on our Cech Complex, we see that $|ker(\delta^1)| = |C^1| = 4$. Our cohomology groups are $H^0(U;R) = ker(\delta^0)/im(\delta^{-1}) = \mathbb{R}^{|ker(\delta^0)| - |im(\delta^{-1})|} = \mathbb{R}^{1-0} = \mathbb{R}^1$ and $H^1(U;\mathbb{R}) = ker(\delta^1)/im(\delta^0) = \mathbb{R}^{|ker(\delta^1)| - |im(\delta^0)|} = \mathbb{R}^{4-3} = \mathbb{R}^1$. We see then, that using n-cubes produced the exact same result as when we used ϵ-balls to generate our cover.

At first glance, it seems unnecessary to use n-cubes when ϵ-balls are available. Well, issues arise if we wish to automate finding cohomology groups via Cech cohomology. When working with a data set, we cannot intersect our larger cover of ϵ-balls or n-cubes with the data set, as the result will be some cluster of points in the data set, the pairwise intersections of which may be difficult to determine. Instead, we only consider the cover formed by centering ϵ-balls or n-cubes at each point in the data set and determine intersections from there. Intersections using ϵ-balls are easy to calculate: we say $U_1 \cap U_2 \neq \emptyset$ if $||U_2 - U_1|| < 2\epsilon$ - that is, if the distance between the centers of two ϵ-balls is less than twice the radius ϵ. However, intersections of three or more ϵ-balls are difficult to determine. Even if the center of each ϵ-ball is less than 2ϵ away from the others, there is no guarantee for triple intersection, as shown in Figure 13. Using ϵ-cubes eliminates this issue, though we must change how we determine intersections. Recall that our ϵ-cubes are comprised of open intervals. To check if two open intervals (a,b) and (c,d) intersect we check to see if $a < d$ and

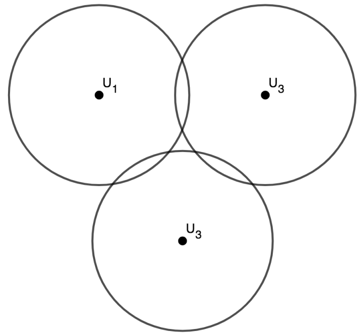


FIGURE 13. A case where each pair of -balls intersect without a triple intersection

$c < b$. Through the use of the following theorems, we can check for any finite number of intersections for any number of -cubes.

Theorem 5.12. *Let $I_1, I_2, I_3, \ldots, I_k$ be open intervals such that $I_a \cap I_b \neq \emptyset$ for all $a, b \in \{1, 2, 3, \ldots, k\}$ where $k \in \mathbb{Z}_+$. Then $I_1 \cap I_2 \cap I_3 \cap \cdots \cap I_k \neq \emptyset$.*

Proof. We begin first by defining the collection of sets $I = \{I_j | j \in \{1, 2, 3, \ldots, k\}\}$. Note that this is a *nite* set, having k elements. Thus, it can be ordered in the dictionary order by $\inf(I_j)$ and $\sup(I_j)$ of each interval respectively. That is, with this ordering of I, for each pair of intervals I_i and I_j we have $I_i < I_j$ when $\inf(I_i) < \inf(I_j)$ or when $\inf(I_i) = \inf(I_j)$ and $\sup(I_i) < \sup(I_j)$. By labeling $a_i = \inf(I_i)$ and $b_i = \sup(I_i)$ for each $i \in \{1, 2, 3, \ldots, k\}$, it follows that $a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_k$ Since we have $I_j \cap I_k \neq \emptyset$ for all $j \in \{1, 2, 3, \ldots, k\}$, applying the previous theorem gives us $a_k < b_j$. Therefore $a_k < b_1$ and $a_k < b_2$ and $a_k < b_3$ and $\ldots$ and $a_k < b_k$. Consider the set $B = \{b_1, b_2, b_3, \ldots, b_k\}$. Note that a_k is a lower bound of B because, as we have shown, $a_k < b_i$ for all $b_i \in B$. Also note that each $b_i \in \mathbb{R}$, so B is a finite, nonempty subset of $\mathbb{R}$. It follows, then, that B must necessarily have a greatest lowerbound, $b \in \mathbb{R}$. We now focus our attention on the interval (a_k, b). Our goal now is to show that $\emptyset \neq (a_k, b) \quad (I_1 \cap I_2 \cap I_3 \cap \cdots \cap I_k)$. Since b is the greatest lower bound of B, and a_k is a lower bound of B, we know that $a_k \quad b$. We know that $a_k \neq b$ because $a_k \notin B$, but since B is a finite set, $b \in B$. Therefore we have the interval $(a_k, b) \neq \emptyset$. Let s be an element of the interval (a_k, b). Then it follows that $a_k < s < b$. But as we have shown earlier, $a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_k$, and since b is a lower bound of B, $b \quad b_i$ for all $b_i \in B$. Therefore we see that $a_1 < s < b_1$ and $a_2 < s < b_2$ and $\ldots$ and $a_k < b_k$. Thus, $s \in I_i$ for all $i \in \{1, 2, 3, \ldots, k\}$ and $s \in (I_1 \cap I_2 \cap I_3 \cap \cdots \cap I_k)$. More importantly, this means that $\emptyset \neq (a_k, b) \quad (I_1 \cap I_2 \cap I_3 \cap \cdots \cap I_k)$, which shows that the intersection of all of our open intervals is nonempty, exactly as we set out to prove.

The results of Theorem 5.12 are useful, but only for 1 dimensional n-cubes (open intervals). Obviously, this will not suffice for a general algorithm. Fortunately, we can reduce any number of n-cubes to a product of intersecting open intervals by proving a certain property pertaining to Cartesian Products.

Proposition 5.13. *Suppose we have k-many Cartesian Products $A = A_1 \quad A_2 \quad A_3 \quad \cdots \quad A_n$, $B = B_1 \quad B_2 \quad B_3 \quad \cdots \quad B_n$, $\ldots$, $K = K_1 \quad K_2 \quad K_3 \quad \cdots \quad K_n$. Then $(A \cap B \cap \cdots \cap K) = (A_1 \cap B_1 \cap \cdots \cap K_1) \quad (A_2 \cap B_2 \cap \cdots \cap K_2) \quad \cdots \quad (A_n \cap B_n \cap K_n)$.*

Proof. Our goal will be to show equality by proving $(A \cap B \cap \cdots \cap K) \quad (A_1 \cap B_1 \cap \cdots \cap K_1) \quad (A_2 \cap B_2 \cap \cdots \cap K_2) \quad \cdots \quad (A_n \cap B_n \cap K_n)$ and $(A \cap B \cap \cdots \cap K) \quad (A_1 \cap B_1 \cap \cdots \cap K_1) \quad (A_2 \cap B_2 \cap \cdots \cap K_2) \quad \cdots \quad (A_n \cap B_n \cap K_n)$.

Let $x = (x_1, x_2, x_3, \ldots, x_n) \in (A \cap B \cap \cdots \cap K)$. Then $x \in A$, $x \in B$, $\ldots$, $x \in K$, which means $(x_1, x_2, x_3, \ldots, x_n) \in (A_1 \quad A_2 \quad A_3 \quad \cdots \quad A_n)$ and $(x_1, x_2, x_3, \ldots, x_n) \in (B_1 \quad B_2 \quad B_3 \quad \cdots \quad J_n)$ and so on. Thus, $x_i \in A_i$ and $x_i \in B_i$ and so on for each $i \in \{1, 2, 3, \ldots, n\}$. Since $x_i \in A_i$ and $x_i \in B_i$ and so on, $x_i \in A_i \cap B_i \cap \cdots \cap K_i$. That is, $x_1 \in (A_1 \cap B_1 \cap \cdots \cap K_1)$, $x_2 \in (A_2 \cap B_2 \cap \cdots \cap K_2)$, $x_3 \in (A_3 \cap B_3 \cap \cdots \cap K_3)$, and so on, which means $(A \cap B \cap \cdots \cap K) \quad (A_1 \cap B_1 \cap \cdots \cap K_1) \quad (A_2 \cap B_2 \cap \cdots \cap K_2) \quad \cdots \quad (A_n \cap B_n \cap K_n)$.

Now we show that $(A \cap B \cap \cdots \cap K) \quad (A_1 \cap B_1 \cap \cdots \cap K_1) \quad (A_2 \cap B_2 \cap \cdots \cap K_2) \quad \cdots \quad (A_n \cap B_n \cap K_n)$. Let $y = (y_1, y_2, y_3, \ldots, y_n) \in (A_1 \cap B_1 \cap \cdots \cap K_1) \quad (A_2 \cap B_2 \cap \cdots \cap K_2) \quad \cdots \quad (A_n \cap B_n \cap K_n)$. We know that $y_i \in A_i$ and $y_i \in B_i$ and so on for each $i \in \{1, 2, 3, \ldots, n\}$, meaning $y_1 \in A_1$ and $y_1 \in B_1$ and so on. But note that $y_1 \in A_1$ and $y_2 \in A_2$ and $y_3 in A_3$ and $\ldots y_n \in A_n$ means that $y \in A$. We see quite easily that this will be true for all k Cartesian Products. Thus, $y \in (A \cap B \cap \dots \cap K)$ and $(A \cap B \cap \dots \cap K) \subseteq (A_1 \cap B_1 \cap \dots \cap K_1) \times (A_2 \cap B_2 \cap \dots \cap K_2) \times \dots \times (A_n \cap B_n \cap K_n)$. Therefore, $(A \cap B \cap \dots \cap K) = (A_1 \cap B_1 \cap \dots \cap K_1) \times (A_2 \cap B_2 \cap \dots \cap K_2) \times (A_3 \cap B_3 \cap \dots \cap K_3) \times \dots \times (A_n \cap B_n \cap \dots \cap K_n)$.

Lastly, we use Theorem 5.12 and Proposition 5.13 in conjunction to prove a stronger statement of Theorem 5.12, which allows us to check for any finite number of intersections of n-cubes of any dimension with ease.

Theorem 5.14. *Suppose we have k-many n-cubes, $A = A_1 \times A_2 \times \dots \times A_n$, $B = B_1 \times B_2 \times \dots \times B_n$, $\dots$, $K = K_1 \times K_2 \times \dots \times K_n$, where any pair of intervals from any n-cube intersect. Then $A \cap B \cap \dots \cap K \neq \emptyset$*

Proof. This is a rather simple proof thanks to Proposition 5.13 and Theorem 5.12. We have $A \cap B \cap \dots \cap K = (A_1 \cap B_1 \cap \dots \cap K_1) \times (A_2 \cap B_2 \cap \dots \cap K_2) \times \dots \times (A_n \cap B_n \cap \dots \cap K_n)$. Since we know that any pairing of any $A_i, B_i, \dots, K_i$ will be nonempty, can apply Theorem 5.12 to each component of $A \cap B \cap \dots \cap K$. Thus, $A_1 \cap B_1 \cap \dots \cap K_1 \neq \emptyset$, $A_2 \cap B_2 \cap \dots \cap K_2 \neq \emptyset$, and so on. From this, we know that for each component $A_i \cap B_i \cap \dots \cap K_i$, there exists some element $x_i \in \mathbb{R}$ which lies in the intersection. If we take all of these components as a single element $(x_1, x_2, \dots, x_n)$, we see that this element lies in the Cartesian Product $(A_1 \cap B_1 \cap \dots \cap K_1) \times (A_2 \cap B_2 \cap \dots \cap K_2) \times \dots \times (A_n \cap B_n \cap \dots \cap K_n) = A \cap B \cap \dots \cap K$. Therefore, $A \cap B \cap \dots \cap K \neq \emptyset$.

By Theorem 5.14, we see that in order to check for k-fold intersections of any-dimensional n-cubes, we only need to check that each pair of components of the n-cubes k-fold intersect. That is, we check that the open intervals of each component of our n-cubes will k-fold intersect in $\mathbb{R}$, and from there we can determine that the n-cubes themselves k-fold intersect.

Recall that our ϵ-cubes will change size depending on ϵ. Given two open covers U and V for a data set consisting of ϵ-cubes and $\epsilon/2$-cubes respectively, it follows that the latter cover will be a refinement of the former. However, increasing the size of ϵ increases the likelihood of higher intersections, and as such the cohomology groups may differ. However, the refinement induces a chain map $(C; V) \to (C; U)$, which in turn by Theorem 2.24 induces a map $H^p(V; R) \to H^p(U; R)$. In other words, we can track the cohomology groups and determine which holes "disappear" based on the increasing size of the cover. This process is referred to as persistent homology, since we see which homology groups "persist" or stay the same as the size of our cover increases.

Remark 5.15. Lastly, there is the matter of the Vietoris Rips Complex, and its relationship to the Cech cohomology using cubes. Both processes require open covers of a topological space X, chain complex induced by the Rips Complex can only be done on a *metric space* X. Intersections in the Vietoris Rips Complex are determined solely by the distances between each open set. The case where three ϵ-balls overlap pairwise, but not all together, is considered a triple intersection in the Vietoris Rips Complex, since the center of each ϵ-ball is less than 2ϵ. However, if we consider the Vietoris Rips Complex with ϵ-balls and the Cech cohomology with ϵ-cubes, we find that both of the generated chain complexes are the same - but only if we change our ϵ to some ϵ' when switching from balls to cubes.

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